Onager Physics

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Introduction

The Onager is a medieval catapult powered by a single bundle of twisted ropes. Very little has been published (1,2) on the physics involved in the design of the device, which has done almost entirely by empirical methods. It would therefore be of interest to apply some simple physical analysis to see what can be done to increase our understanding of this seemingly simple device.

Our approach will be to devise a couple of simple models for the device, and to analyze them using elementary physical considerations to derive analytical expressions for the torque on the beam as a function of the Onager dimensions and the configurations of the skein. We hope to thereby arrive at a level of explanation that encompasses enough of the mechanism to be assured that the essence of the machine is captured, but not so complex that it is useless to all but the professional mathematician.

The ropes in the bundle or skein can be configured in a myriad of ways, none of which appear to have been described before, or even named. Therefore the examination of the properties of the skein will be the central focus of this paper. We want to understand where the torque comes from, where the energy goes, describe some measures of efficiency of the device and the trade-offs involved in design. We want to have some rational method for choosing the material that will give the best performance and how to best expoit those characteristics.

We would also welcome some general guidlines on the scaling properties of the Onager: how does the energy in the skein scale with the linear dimensions of the parts? How does the range of the projectile scale? How should we scale the parts so they don't break?

The dynamics of the throw when the beam is released will be treated separately, hopefully, in another paper in this series.

The Geometry

Fig. 1 shows two schematic views of an Onager: one from the side, and the other from above. The twisted skein or bundle of twisted ropes are separated into two parts by a beam, which is held tightly under tension from the bundle. The tension in the skein is produced by means of one (or two) rotating capstans through which the skein is guided.

When the Onager is cocked, the beam is held by some trigger mechanism as shown, the projectile placed in the sling, then released. During the throw, the beam rotates very quickly to a vertical position, where it is stopped by the padded stopping bar. When the beam rotates CCW, the sling with the projectile rotates outward to the left. At some point in the rotation, the sling opens up, releasing the projectile, which flies off to the right, hopefully busting something not on the Onager.

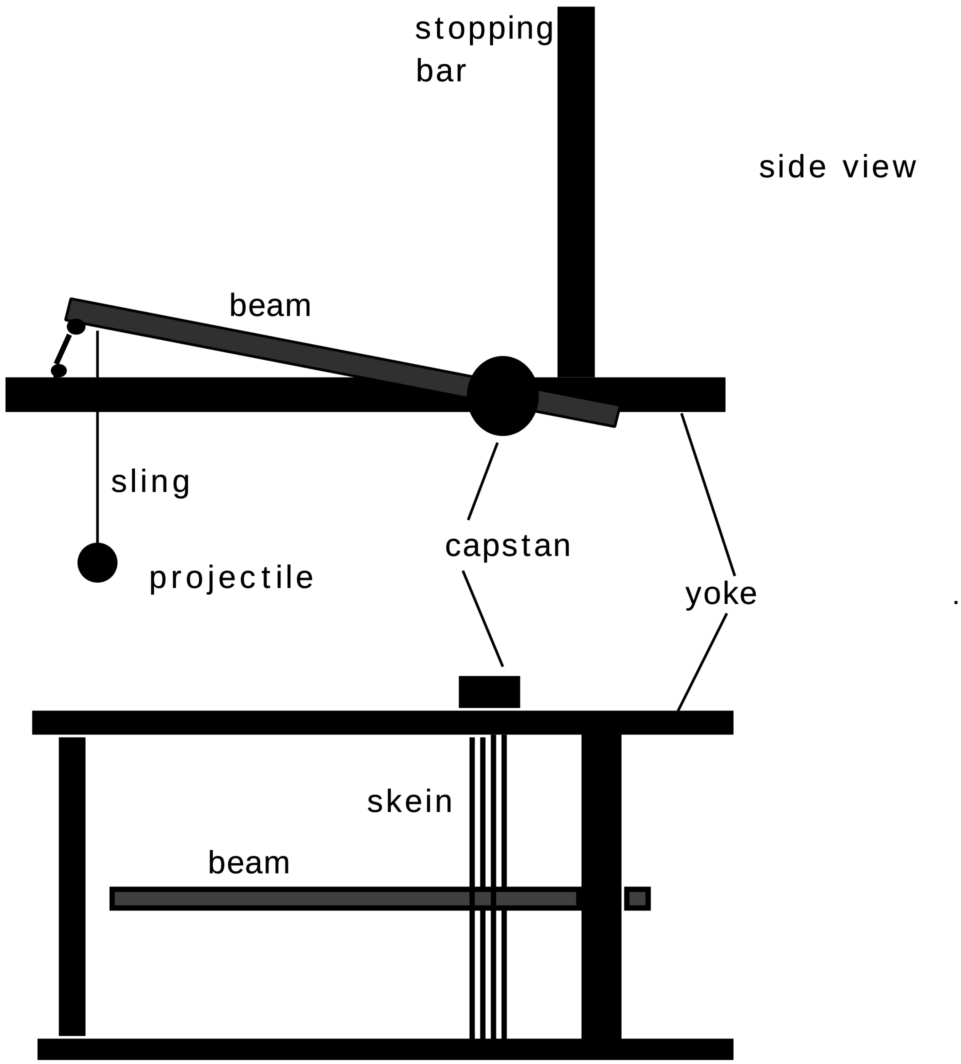

Fig. 1. The Onager and its parts defined.

The Torque on the Beam

The object of central interest in our models is, of course, the torque imparted to the beam by the rope bundle. We'd like to know, for example, how the torque varies with the position of the beam, the tension in the ropes of the bundle, and the dimensions of the bundle, the yoke and the beam. We'd like to be able to use this to look at the forces tending to break the parts of the Onager in order to make its design and construction more rational.

First, we need to review some simple stuff about what torque is, and how it arises in the Onager.

Torque is the quantity that makes the beam rotate, so it is rather important in what follows to have a good understanding of it. Appendix I contains a general discussion of torque and how to calculate it for readers who need a little bit of brushing up on their high school physics.

Tension in a Rope

The force that creates the torque in the Onager is the tension of the ropes in the bundle. We think of a flexible string. The force communicated by a stretched flexible string is called the tension. It can act only along the direction of the string. If we connect two bodies by a stretched string, then the tension acts in a direction tending to draw the two bodies together. Flexible strings cannot provide thrust, a force tending to separate the two bodies. Consider a simple example:

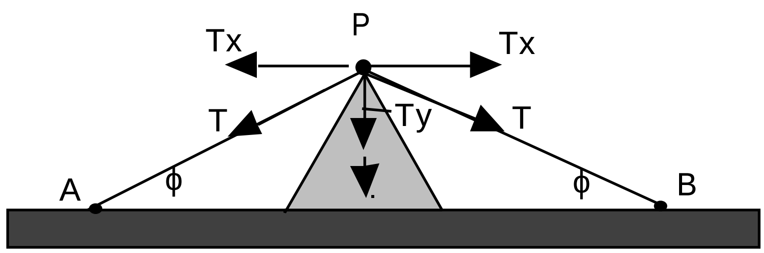

Fig. 2 Example of two strings under a tension T pressing a triangular beam into a base.

Here a string is attached to a base at A, then to the apex, P, of a beam that has a triangular cross-section. Another string goes from P to B that is under the same tension T. The string makes an angle ϕ with respect to the base. The Tensions can be resolved in the horizontal and vertical directions as shown. We can easily see that force acting at the apex in the horizontal direction is zero because the two components cancel. In the vertical direction, the two components of the tension add, so the force pressing the beam into the base is just $F = 2\ T \cos(\pi/2-\phi) = 2\ T \sin(\phi)$. If we replace the two strings with a single string passing over the apex, under the same tension, the force pressing the beam into the base would be unchanged.

The force pulling on A is also a vector along the string, with a magnitude T, but with a direction opposite to that shown. The force tending to draw the two points A and B together is then $2\ T \cos\phi)$.

One complication that must be faced in dealing with the Onager is that the skein or rope bundle is made from ropes that have a non-zero diameter, unlike the strings of the usual mechanical models. In the system shown above a more realistic drawing of the actual set-up would be something like this:

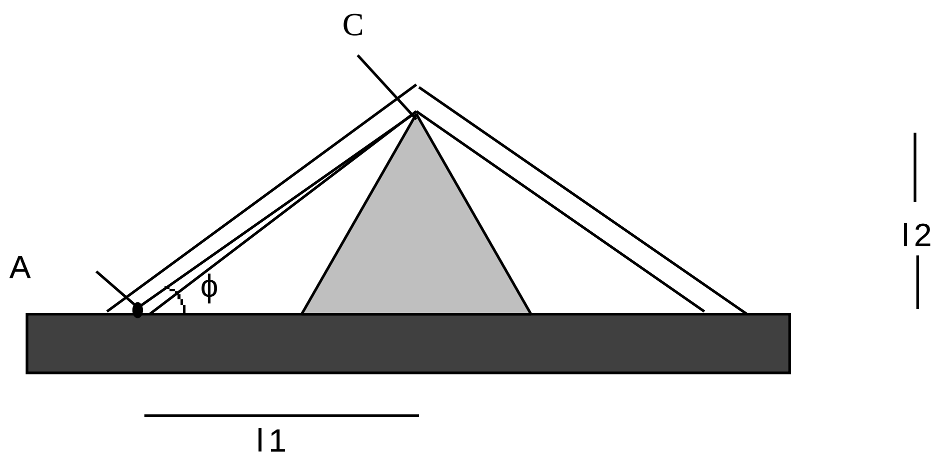

Fig. 3. The rope is not a string!

Here in Fig. 3, the string is replaced by two parallel lines representing the outer limits of a rope with a certain definite diameter. A real rope is not fastened at a single point, nor does it contact a surface at a single point, unlike the physicist's string. A rope is quite a complicated object--it consists of overlapping fibers that transmit forces between themselves by means of a their very high mutual friction. Ropes have a certain stiffness and anisotropy in their properties. When placed under tension, their diameter shrinks a little. They are extensible and compressible. And so on...

To make a tractable model the Onager, however, we still need to extract the right essential properties of the ropes comprising the skein. It will become clear later in our discussion that the most important factor is the finite diameter of the ropes needs to be retained.

Where then are we to draw our force vector representing the tension in the rope? Although other choices might be made, for the purposes of our model, we will explicitly assume that the force vector representing the tension is drawn from the point of contact, C, to the center of the point of attachment of the rope at A. This is an approximation which will much simplify our analysis.

Note that in this case, with the dimension l1 and l2 as shown, the force pressing the beam into the base is

$$2\ T \sin(\varphi) = 2\ T\ l2/\mathrm{sqrt}(l2^2+l1^2)$$

which is useful in comparing results to be obtained (vide infra).

Dimensional Analysis of the Onager

A useful method to understand the properties of the Onager is to consider how they scale with changes in dimension of the parts. Does one expect, for example, an Onager that is twice as big to be twice as powerful? Is the torque on the beam twice as great? Will it throw a given weight twice as far?

The parts of the Onager have been previously described: the distance between the attachment points, 2 l1, the thickness of the beam, 2 l2, the diameter of the rope, 2s, and the thickness of the attachment points, 2t. The tension in the rope, T, is the principle force involved.

We assume that the frictional forces are negligible to a first approximation, and base our discussion on this. An Onager scaled by a factor of k, say, would have a distance between the yoke of 2 k l1, beam thickness 2 k l2 and a rope diameter 2 k s.

A simple dimensional analysis (or, by inspection) of the problem of the tension then shows that the torque would be given by

$$\tau = C\ f(l1/s, l2/s, t/s)\ s\ T$$

where C is some constant, and f is some undetermined function. The torque is proportional to the tension, which should come as no surprise.

The energy in the Onager is contained in the rope, and is proportional to its volume. Scaling the Onager by a factor k, would increase the volume of the rope by a factor k l1 (k s)^2 or by k^3. That is, doubling the dimensions of the Onager would increase the energy in the bundle by 8 . So the scaled Onager would have an energy scaled by k^3.

Since the range of a projectile of mass m with initial kinetic energy E is (vide infra) proportional to E/m, we'd expect the range for a given mass to increase by a factor of k^3 too. The range of a given mass for an Onager twice as large would be greater by a factor of 8. Or a mass twice as great would go 4 times as far...

The energy in the Onager scales with the 3rd power of its dimensions. For comparison, the trebuchet scales with the 4th power of its dimensions (volume of counterweight x height).

In scaling up an Onager, an important question is "How should we scale the yoke? " It can be shown (see ref 3) that the maximum stress that a beam with a square cross-section of thickness, r, supported in the middle, with the supports a distance L apart, under a weight P scales as PL/r^3. This scales as k^2 k/k^3 = 1. So a scaled Onager, with the thickness of the yoke increased by k, should be OK. It won't break.

The energy in the beam crashing into the stop will be proportional to the energy in the skein, which, as we have seen, scales as the fifth power of the dimensions. This energy is dissipated into a volume of the beam. If the energy deposited per unit volume of the beam exceeds a certain value, the beam will break or be crushed. Therefore to scale up an Onager by a factor of k requires that the volume of the beam in which the energy is dissipated be scaled by k^3. If this volume is proportional to the thickness of the beam cubed (square cross-section), this requires that the thickness be increased by a factor of k^(3/3). So doubling the size of the Onager requires that the thickness of the (square cross-sectional) beam be increased by a factor of 2. .

A Two Stranded String Model

We want now to consider a string (zero radius) model first to introduce some fundamental ideas about how to picture the Onager configuration, set up our coordinate system and notation, and so on. This is the "two stranded string model". We can easily modify the configuration shown in Fig. 2 to a configuration more like an Onager:

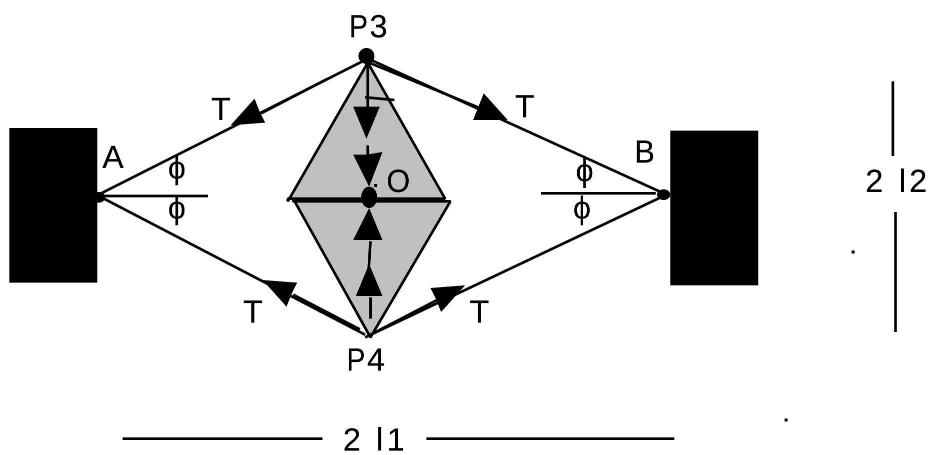

Fig. 4. A beam with diamond cross-section being compressed by two strings.

Here A-P3-B are one strand of string (with zero radius) fastened at A and B, and A-P4-B is another strand fastened to the beam (which now has a diamond cross-section) at P4. The points of attachment A and B are fixed to the ground and can't move. The apexes are a distance 2 l2 apart. A-P4 -B-P3 are set-up to be all in one plane. The base is gone and the beam is supported only at P3 and P4.

Then the force exerted by the string at P4 is now up and has the same magnitude as the force at P3. The forces acting at P3 and P4 tend to compress the beam in the vertical direction. As long as all of the tensions are equal, it is clear, and easy to show, that there is no torque about O. This model is not good enough--something is left out.

One might imagine that a better model would have P3 and P4 displaced into and out of the paper by some (equal) amount. Would a torque on the beam be produced then? It is not hard to set up a coordinate system, get the vectors of the tensions, and calculate the torque on the beam analytically as the cross product of vectors from O to the points P3 and P4. You'd again get a torque of zero. Displacing the points of application of the tensions on the beam still leaves something out.

An easier way to see this is by a consideration of the couples produced by the four forces. Note that P1, P2, P3 and P4 are still co-planar. At P3 the forces have a resultant in that plane that points downward towards P4. At P4, the two forces also have a resultant that lies in that plane, and points upward towards P3. That is, the two resultant forces are anti-parallel. However, they both act along the same line P3-O-P4, which means that their moment arm is zero, so the couple has a zero magnitude, and there is no turning force acting on the beam.

This configuration of the string looks pretty much like an idealization of the bundle when the Onager is cocked, and so it may come as a surprise that the torque is zero. So where does the torque in a real Onager come from?

One could suppose that the torque is produced by an asymmetry in the forces or the positions of the points of application of the forces at P3 and P4. A torque would certainly be produced, but this is not a very elegant route to take, and would be quite ad-hockish. The Onager has a symmetry to it that the model should preserve.

What appears to be required, as we pointed out before, is to introduce the idea that the rope (or sling bundle) radius has a definite non-zero value. This means that the points at which the rope separates, P1 and P2 are not really single points.

There is no torque induced in the Two Stranded String model.

Multi-Stranded Open Twisted Model

One possibility involves a skein that is highly twisted, which contacts the beam at a pair of points. Fig. 8 shows this idea embodied in a small model. The cylindrical white object on the left is a capstan that holds a straight bar of metal rod, parallel to the yoke, that the rope can be wound around. This configuration is arrived at by imparting a few twists in the skein by means of the capstan, sliding them over to the right, inserting the beam, then winding the capstan an equal number of turns in the same direction. The skein shown is composed of four strands that separate at a point some distance away from the beam, creating an opening, or gap there. We'll call this the "multi-stranded open twisted" model for the Onager skein.

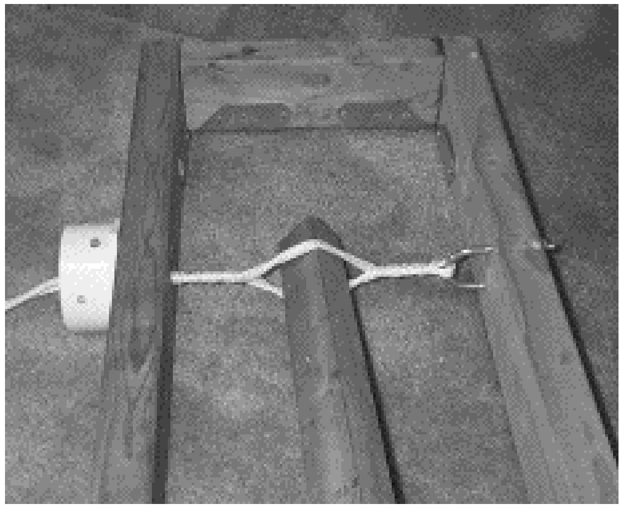
Fig. 5. The open twisted bundle configuration when cocked.

To model this idea, referring to the previous figure, the point at P1 just before the untwisting of the rope bundle can be left where it is, and we can introduce two new points, P5 and P6, which is the start of the rope's straight line trajectory to the points of attachment to the beam. The region of rope between P1 and P3 can have some complicated shape, but we needn't worry about this. Similarly, it appears to be safe enough to leave the points of attachment of the rope to the beam where it is, and as a single point. To see this model, consider the diagram of the Onager with the beam now turned up, and the beam omitted from the drawing for greater clarity.

Open Twisted Onager Force Diagram

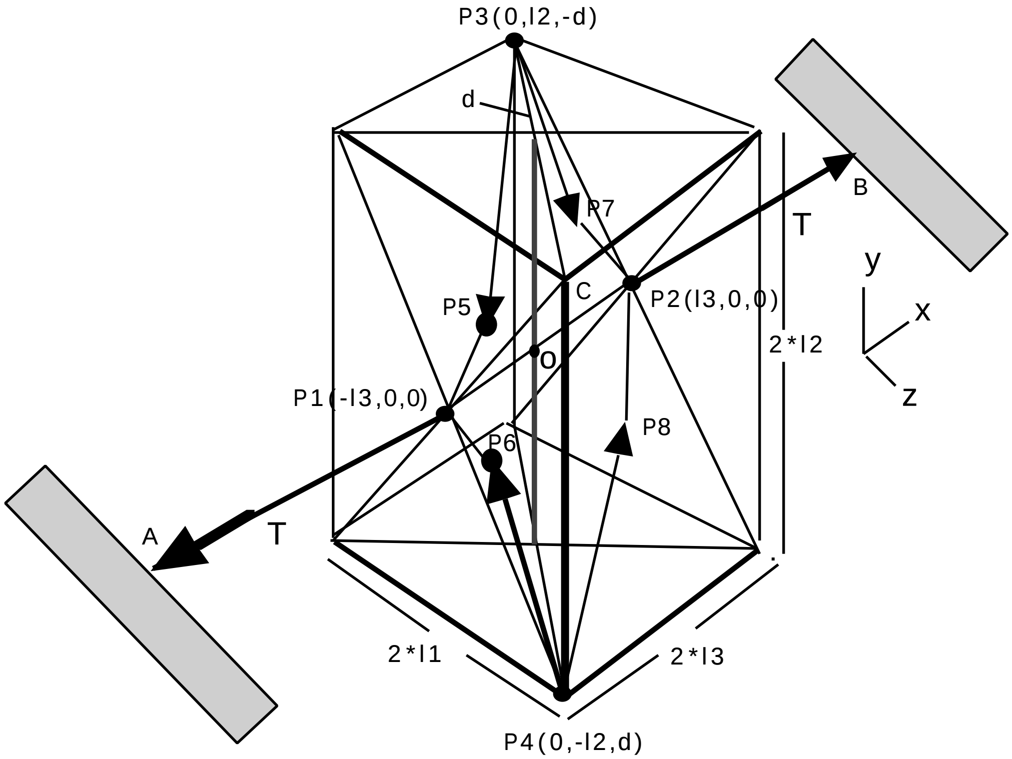

Fig. 6. The open twisted Onager with points of attachment A and B, points of contact P3 and P4 and with the beginning of the each untwisted strand displaced by small amounts from the twisted bundle at P1 and P2.

The rope is fastened to the yoke of the Onager at A and B; it separates into 2 strands at points P1 and P2 and pulls on the beam at the points of contact, P3 and P4.

The origin of the coordinate system is midway along the line P1-P2, designated by "O"; P1 is on the front left face of the parallelepiped, P2 is on the rear right face.

P3 and P4 are the points where the rope contacts the beam (not shown) of the onager and the force coming from the rope acts at that point. The origin is along the central axis of the beam.

Clearly, the point at which each strand becomes free from the other(s) has an x-coordinate that falls between P1 and P2. The transition region, where the unwinding of the bundle occurs (for the upper strand it is the segment P1-P5. The points P5 and P6 are the points where a strand begins its (straight line) path to the point of contact with the beam.

Thus, P6 and P5 are both inside the parallelepiped. To summarize, the upper strand of the rope follows the path A-P1-P5-P3-P7-P2-B and the lower strand follows the path A-P1-P6-P4-P8-P2-B.

P5 is above and inward from P1: P5(-l3+d1,d2,-d3) where d1, d2, and d3 are positive and, clearly, are lengths of order of the radius of the rope bundle.
P6 is below and inward from P1: P6(-l3+d1,-d2,d3).

P3 and P4 are in the yz plane; P3, P4, O and C are all coplanar.
P3 has coordinates (0,l2,-d) and P4 has coordinates (0,-l2,d), where d is Sqrt(l2^2+l3^2) only if the parallelepiped is a rectangular parallelepiped.

A vector from P3 to P5 is (-l3+d1,d2,-d3)-(0,l2,-d) = (-l3+d1, d2-l2, -d3+d) ≡ v_{tl}
A vector from P4 to P6 is (-l3+d1,-d2,d3)-(0,-l2,d) = (-l3+d1, -d2+l2, d3-d) ≡v_{bl};
v means vector, tl means "top left" strand, bl means bottom left strand.

The length of each of these vectors is Sqrt((-l3+d1)^2 + (d2-l2)^2 + (-d3+d)^2) = v, so a unit vector from P3 to P5 is v_{tl}/v and a unit vector from P4 to P6 is v_{bl}/v.

The magnitude of the tension in the half strands of the rope going from P3 to P5 is T', and that going from P4 to P6 is assumed, as before, to be the same. Therefore the vector of the force pulling from the left at P3 is T' v_{tl}/v and the vector of the force pulling on P4 from the leftmost half-strand is T' v_{bl}/v.

The vector from the origin, O, to the point where the force coming from the top left half strand is (0,l2,-d)-(0,0,0) = (0,l2,-d) ≡ r_{tl}. The torque, τ, about the origin, due to the upper left half strand, is the cross product of r and the force: $\tau_{tl} = r_{tl} \times T' v_{tl}/v$, and similarly for the lower left half strand. Adding all four tourqes from the half strands will give the torque on the beam.

The calculation is easily done, and the result is that the total torque on the beam is found to be

$$\tau = \frac{4(d\,d2 - d3\,l2)T'}{\sqrt{(d-d3)^2 + (d2-l2)^2 + (d1-l3)^2}}\,\mathbf{i}$$

where i is a vector in the x-direction, and T' is the tension in each half strand. Note that when d3 = d2 = d1 = 0, we recover our result from before, that the torque is zero. While, pretty general, this formula is not terribly useful as it stands.

When d3 = d2 = d1 = rb (effective radius of the bundle) then this simplifies to

$$\tau = \frac{4(\frac{d}{rb} - \frac{l2}{rb})\ rb\ T'}{\sqrt{(1-\frac{d}{rb})^2 + (1-\frac{l2}{rb})^2 + (1-\frac{l3}{rb})^2}}\ \mathbf{i},$$

where now all of the lengths have been scaled by the effective bundle radius, rb.

It is not hard to show that the tension in the half strands, T' is given by

$$T' = \frac{\sqrt{d1^2 + d2^2 + d3^2}}{2\ d1}\ T,$$

where T is the tension in the twisted strands. In the case where d3 = d2 = d1 = rb, we get

$$T' = \frac{\sqrt{3}}{2}\ T$$

so the torque on the beam finally becomes

$$\tau = \frac{2\sqrt{3}(\frac{d}{rb} - \frac{l2}{rb})\ rb\ T}{\sqrt{(1-\frac{d}{rb})^2 + (1-\frac{l2}{rb})^2 + (1-\frac{l3}{rb})^2}}\ \mathbf{i}$$

The equation satisfies our physical intuition in several respects. The torque is proportion to the tension, as expected. It has only an x-component, which means the beam is not being subject to slewing forces tending to drive it to one side or the other. The factor (d-l2) is a surprise however. This means that when d = l2 the torque is zero--it changes signs there! In the (nominal) case where d=0, the magnitude of the torque is

$$\tau = \frac{2\sqrt{3}(\frac{l2}{rb})\ rb\ T}{\sqrt{(\frac{l2}{rb} - 1)^2 + (\frac{l3}{rb} - 1)^2}}$$

Since l3, half the distance between the points of attachment, does not appear in the numerator, it is easy to see that the smaller l3 is, the larger the torque. The minimum value l3 can take, of course, is l2, halt the width of the beam. A special case is when the bundle separation point comes right up to the (diamond shaped) beam. We'll can call this the "closed twisted bundle" model. Then l3 = l2 and the torque is

$$\tau = \frac{\sqrt{6}\,(\frac{l2}{rb})\ rb\ T}{\sqrt{(\frac{l2}{rb}\ -1)^2}}$$

And if l2/rb>>1 then the torque is

$$\tau \approx \sqrt{6}\ rb\ T.$$

Note that in this model rb < l2 because otherwise, the upper rope would pass over the beam without contacting it.

Fig. 7 shows the torque as a function of effective bundle radius, when l2 is 10. The torque increases faster than linearly, showing that a large bundle radius has some advantages, other things being equal.

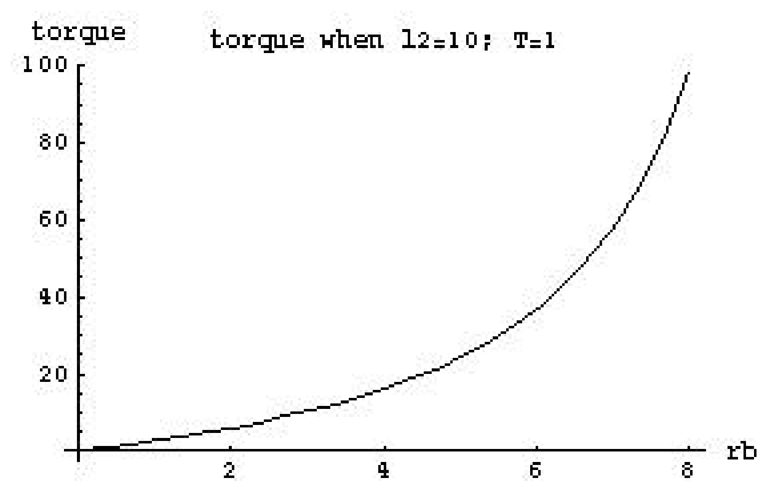

Fig. 7. The torque-effective bundle radius relationship for an Onager with l2=10, scaled with respect to the tension.

Tension in the Twisted Bundle Model

We need to better understand, now, the general origin of the tension in the Onager bundle when the skein is highly twisted, and how it depends upon the rotation of the beam and the properties of the rope.

The open twisted model involves just the segments from A to P1 and B to P2. Here the ropes in the bundle are twisted around one another in some way. Visualizing a single rope in isolation shows that the twisting might be most readily modeled as a circular helix. As shown in the diagram, a helix has a length L, in which n turns are made. The distance between the turns is the pitch, p, given by L/n. The radius of the helix is r_0. Figure 8 shows a helix with 8 turns.

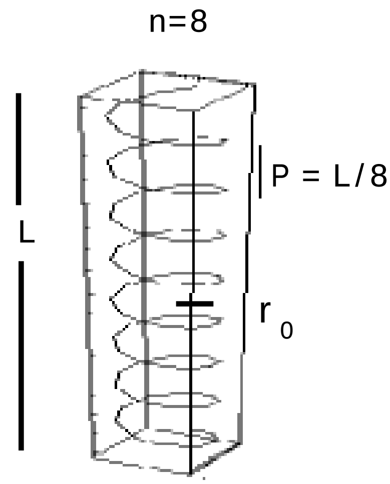

Fig. 8. The definition of the parts of a circular helix.

The length of the space curve for a helix with n turns is

$$S = 2\pi n\sqrt{r_0^2 + (p/2\pi)^2}$$

or

$$S = 2\pi n\sqrt{r_0^2 + (L/2\pi n)^2}$$

When cocking the onager, we are increasing the turns in the helix by a certain amount, Δn. For a first approximation, we can assume that the distance A-P1 doesn't change during the cocking, and the cocking involves changing the number of turns by one quarter of a turn. The length of the rope making up the helix changes therefore changes during cocking by

$$\Delta S = \frac{\pi}{2}\sqrt{r_0^2 + (L/2\pi n)^2}$$

The rope has a characteristic stress-strain curve which can be used to calculate the tension. In the linear part of the curve, the tension in the rope is proportional to how much it has been stretched. That is, T = k Δs, where k is an elastic constant for the rope. During the throw the helix unwinds by a quarter turn: Δn = 0.25, and so the tension, and the torque, both decrease during the throw--but not to zero.
Since

$$\theta = 2\pi / n,$$

the dependence of the tension on the twist is

$$T = k\,\theta \sqrt{r_0^2 + (L/\theta)^2}$$

It should be clear, however, that the bundle is made up of a number of ropes having various values for r_0, and some number of different helices would need to be considered to make the actual calculation of the overall tension, T. This would vary with the structure of the bundle, and how it relates to the beam and the points of attachment. Various assumptions could be made for the configuration in thicker bundles, but the methodology for working out the torques and tension would be basically unchanged.

Parallel Bundle model

Another method of arriving at a twisted bundle involves starting as before with a parallel bundle of ropes. If there are four strands of rope, this is called the "4-stranded simple parallel bundle" and is a simple configuration. In this type of skein the ropes are passed through their attachment in a parallel fashion under tension, then the beam is vertically inserted into the bundle, separating them into two pairs, without inducing any twist or crossing over of the strands. Then the onager is cocked by moving it into the horizontal direction. This is depicted in a model in the next three pictures.

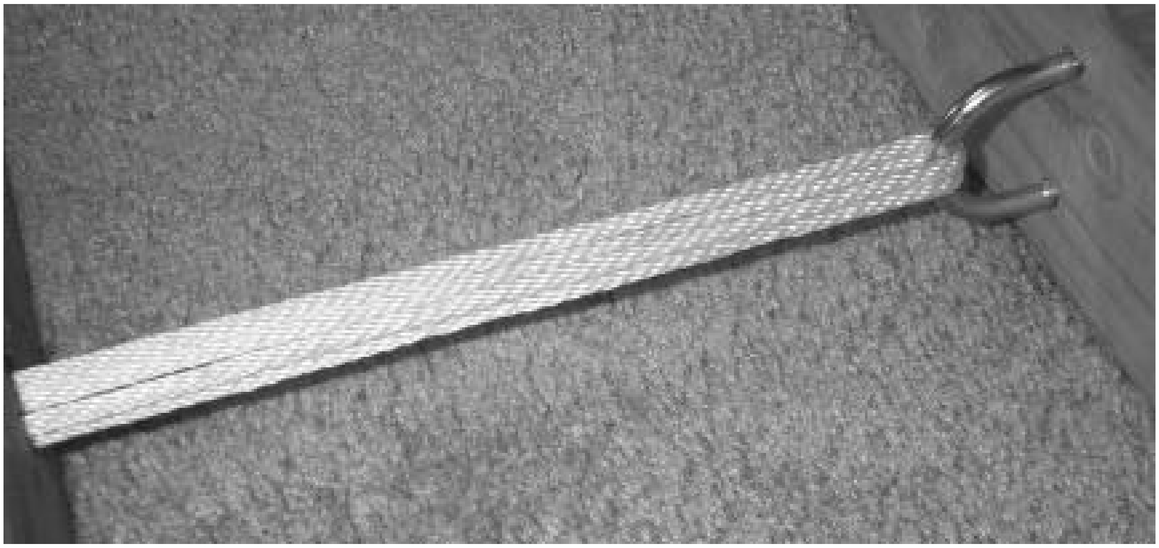

Fig. 9 Four parallel strands--no crossing of ropes or twists.

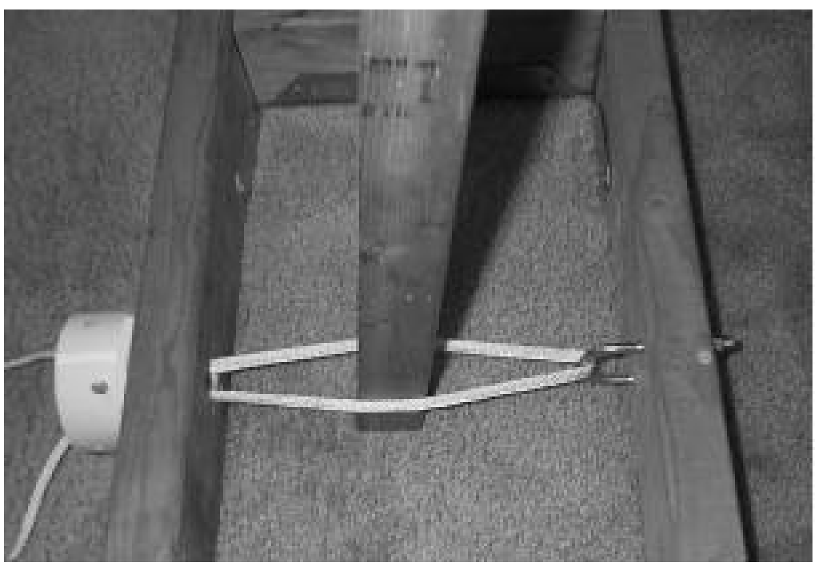

Fig. 10. The beam is inserted vertically separating the bundle--still no twists or crossing in the ropes. The beam is perpendicular to the plane of the yoke.

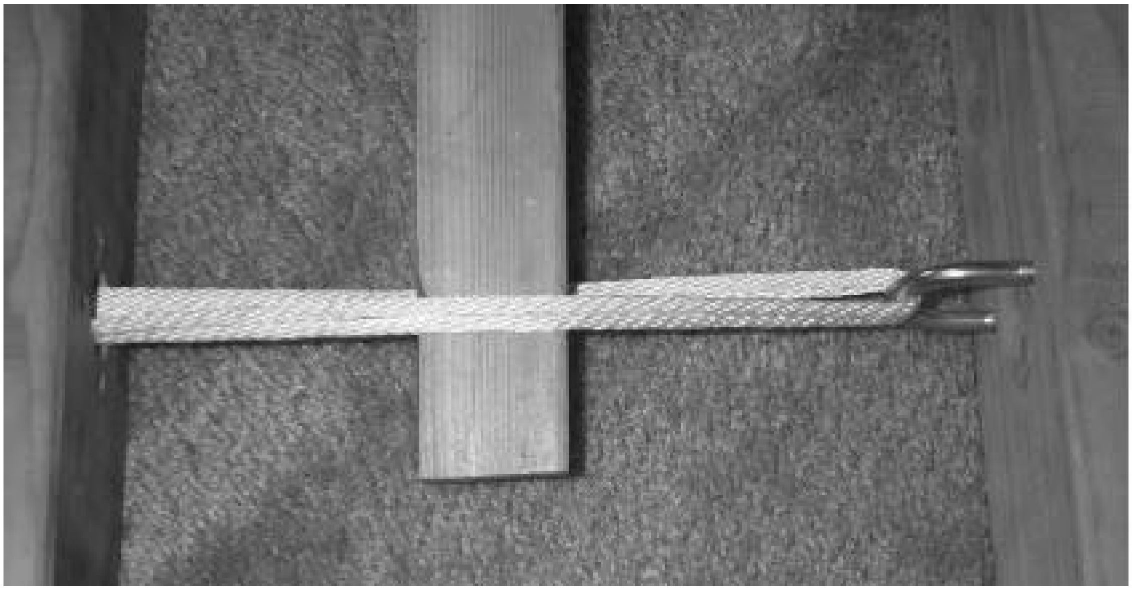

Fig. 11. The beam is cocked by moving it to be parallel to the yoke, imparting a twist in the ropes.

In the first of the pictures, we see that we can adopt a simple nomenclature for the strands. There is an upper strand, in the front nearest the camera. Call this the upper front strand, or "uf" strand for short. Similarly, there is a lower front, upper back, and lower back strand: lf, ub, and lb strands for short. The rope, which is actually one continuous piece enters from the left, then forms the uf, ub, lf, lb strands successively. It is tied off under some initial tension at the left capstan.

When the initial tension in the rope is high, we can observe experimentally that the ropes do not slide along the beam as the beam is moved from the vertical to the horizontal position. It is as if the point of contact of the strands to the beam are actually fastened to the beam. We will assume this to be the case here. This may not be the case generally, for it depends upon the friction between the beam and rope.

In the second picture, which has the beam inserted, the order of the strands are not changed, but the front and back strands are separated in the center by the height (2 l2) of the beam. This is farther than they were separated initially, so the ropes are under greater tension in the second picture than in the first. To get the beam in, a wedge can be employed to widen the opening between the front and back strands.

In the third picture, in which the beam has been cocked by moving it horizontally, the two nearest (front) strands have moved. The uf strand passes over the beam and contacts the beam farther from the camera than the lf rope. Similarly, the ub strand contacts the beam farther from the camera than the lb strand.

We are, of course, interested in the torque created by the cocking of the beam--the movement from the vertical to the back, horizontal position. Therefore we need to get the position of each of the strands where they contact the beam, and where they are fixed at the points of attachment. We then form unit vectors along each of the four strands in the direction of the tension, and perform the cross product of the radius from the origin to the point of attachment on the beam. The sum of the four tourques is the torque on the beam. The torque will be along the x-axis.

Let A be the attachment at the left and B the attachment at the right. The attachments are assumed to have a thickness 2 t so the central axes of the front and back strands at the attachments are separated by t+s, where 2s is the diameter of the rope.

Two Stranded Parallel Bundle Model

The simplest arrangement for the skein would involve just two parallel strands, rather than the four illustrated in the pictures. This is not as strange as it might seem--a thick complicated bundle can be usefully modeled as a single thick rope, after all.

We set up a coordinate system with the origin in the center of the beam at 0, midway between the points of attatchment at A and B. Put the x axis through A-O-B. The z-axis is up, perpendicular to the plane of the yoke. The y-axis is through O perpendicular to the xz plane. It is along the beam when it is horizontal (cocked).

Using the same notation as before, we can write down the coordinates for the points of attachment and contact with the beam and calculate the torque imparted to the beam.

The two strands (f=front, b=back)attached at the left have coordinates

$$Af = (-l1, t+s, 0);$$
$$Ab = (-l1, -t-s, 0);$$

and the two back strands have coordinates at the points of attachment

$$Bf = (l1, t+s, 0);$$
$$Bb = (l1, -t-s,0).$$

The points of contact with the beam when it is cocked (i.e., horizontal) are

$$Pf = (0, t+s, l2);$$
$$Pb = (0, -t-s, -l2).$$

The unit vector from the point of contact to the point of attachment at A for the forward left (=L) strand is then

$$TfL=(Af-Pf)/|Af-Pf|$$

$$= (-l1/m, 0, -l2/m)$$

and similarly for the three other half strands, where the || signifies magnitude. The magnitude of the vector is

$$m = |Af - Pf| = \mathrm{sqrt}(l1^2 + l2^2).$$

The vector from the origin to the point of contact is just Pf for the front strand and Pb for the back strand. Then the total torque, τ, is the sum of torques coming from each half-strand, so

$$\tau = T\,(Pf \times (TfL + TfR) + Pb \times (TbL + TbR))$$

where T is the tension in each half-strand, all assumed to be equal. It is easy to see by symmetry the two terms on the right are equal, so that this can be simplified to

$$\tau = 2\,T\,(Pf \times (TfL + TfR).$$

The sum of the tensions from the left and right strands is

$$T\,(TfL + TfR) = T\,(0, 0, -2\,l2/m)$$

so, finally, the torque on the beam when it is cocked is the cross-product

$$\tau = 2\,T\,(0, t+s, l2) \times (0, 0, -2\,l2/m),$$

or, finally,

$$\tau = (-4\,T\,l2\,(t+s)/m, 0, 0),$$

which shows, as expected, that the torque vector is along the x-axis.

This result can also be readily seen by using the fact that there is a couple involved: the two forces tf and tb are anti-parallel, with a magnitude 2 T x l2/m. The moment arm between them is 2(s+t).

The calculation can also be done using the Mathematica programming language, and is shown for comparison in Appendix II. Comparing the two methods should allow the reader to understand this language well enough to follow along in the calculations for more complex cases to come.

The torque on the beam when cocked is proportional to T, and increases with the beam thickness, both of which is reasonable. It is zero when t and s are both zero, also as expected, based on our earlier discussion of the string Onager.

It is also easy to show that the force pulling on the left attachment at A is

$$F_A = 2\,T\,(l1/m, 0, 0)$$

when the beam is cocked. This is only in, along the x-axis. For a fixed l2 this increases with l1, so it is better to design for a short distance between the yoke arms: they will be less likely to break. Making the beam thicker (thickness = 2 l2) will also decrease the force tending to break the yoke.

Cocking the beam by moving it from the vertical to the horizontal direction will change the lengths of the two strands, and therefore their tension will change too. But how do we calculate the length of a half-strand? Do we measure along the center of the strand, or along the path that our tension takes? The two possibilities give slightly different answers, depending on s and t. There appears to be no easy way to resolve which is the better approximation.

It is slightly easier to calculate the latter possibility, and we'll use this method. The distance from the point of contact to the point of attachment when the beam is cocked is

$$dc = \mathrm{sqrt}((af-pf).(af-pf)) = \mathrm{sqrt}(l1^2 + l2^2).$$

When the beam is upright, the point of contact is at (0, l2, 0), so the corresponding distance is

$$du = \mathrm{sqrt}(l1^2 + (l2 - s - t)^2),$$

which is slightly smaller. Therefore cocking the beam increases the tension in the strands.

Given the initial tension in the strands before the insertion of the beam, and the elastic constant for the rope allows one to readily calculate the tension in each rope before and after cocking.

Torque as a function of angle

In order to get the dynamics of the Onager, we need to obtain the torque on the beam when the beam is at arbitrary angle, θ, with respect to the horizontal axis. When the beam is fully cocked, θ=0 and when it is vertical, θ is π/2.

At first we will concentrate our attention to the forward strand only. The back strand analysis follows easily by symmetry. For the Two Stranded Parallel Bundle Model, when the beam is upright, the point of contact of the forward strand with the beam is at pf = {0, l2, 0}. When the beam is horizontal, it is at {0, t+s, l2}. We need to get pf for an arbitrary angle. Start with the beam in the vertical configuration. We have assumed that the point of contact doesn't move with respect to the beam. Therefore when we rotate the beam toward the horizontal direction the point of contact will move along a circular path in the yz plane with the axis of rotation going through the origin of the coordinate system. The radius of the path will be l2. In the first part of the rotation, therefore, the coordinates of pf will be pf {0, l2 sin(θ), l2 cos(θ)}.

However, this cannot be the case for the entire range of θ because it doesn't have the right value when theta is zero. A diagram shows that when the beam is rotated, the strand comes to a point where it reaches straight across the yoke, and the axis of the strand has the equation y = t+s. The figure makes this clear.

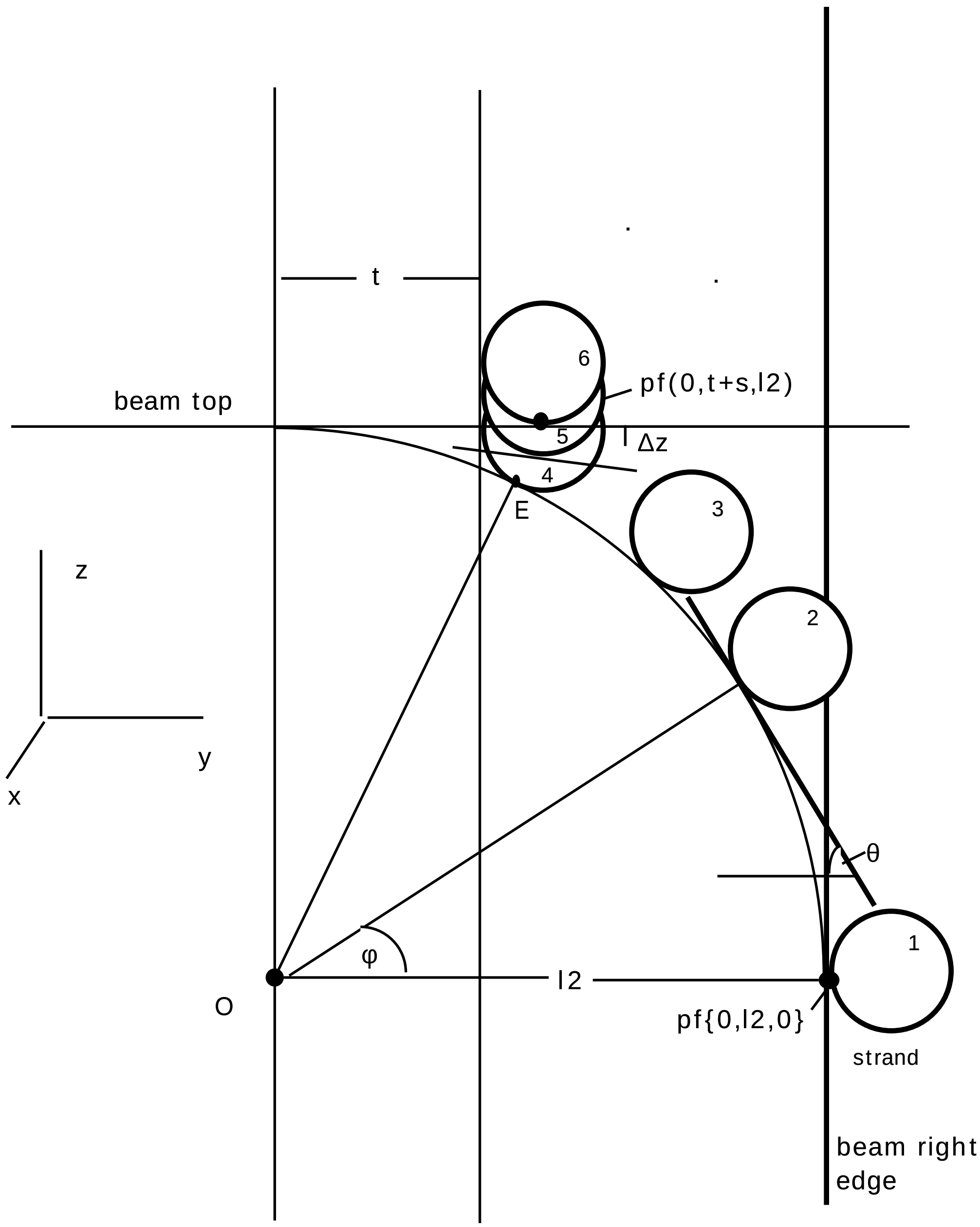

Fig. 13. The movement of the strand is hindered by the width of the point of attachment.

Fig. 13 shows the successive positions of the strand as the beam is rotated from upright to horizontal. The start is at position 1 and the end position when the beam is horizontal is at 6. At some point in the rotation, shown here as position 4, the strand can't move any more to the left because of the thickness of the point of attachment, t, and the finite radius of the strand, s. The point of contact therefore leaves the circle centered at O and travels upward as φ increases.

The upward travel starts when the point of contact is at E, and a little geometry reveals (love that phrase!) that it has coordinates

pf {0, l2(t+s)/(l2+s), l2*sqrt(1-((t+s)/(l2+s))^2)}; $\varphi = \varphi_E$

and the angle where this occurs is φ_E = arc cos((t+s)/(l2+S)).

The coordinates of the point of contact when $\varphi > \varphi_E$ can be shown (more geometry) to be

pf {0, t+s-s*cos(φ), l2/sin(φ)-(t+s- s cos φ)/tan(φ)}; $\varphi > \varphi_E$

Again, in the first part of the rotation the coordinates of pf will be

pf {0, l2 cos(φ), l2 sin(φ)}; $\varphi < \varphi_E$

where $\varphi = \pi/2 - \theta$. Using these expressions we can calculate the torque/tension as a function of angle of rotation of the beam. The expressions are derived in Appendix II, and are

$$\frac{4\, l2\, (s+t)\, \mathrm{Cos}[th]}{\sqrt{l1^2 + l2^2\, \mathrm{Cos}[th]^2 + (s+t-l2\, \mathrm{Sin}[th])^2}},$$

for $\theta > \theta_E$

and for the hindered movement portion of the beam rotation,

$$\Big\{-\Big(\sqrt{2}\,(s+t)\,\mathrm{Sec}[th]\,(-2\,l2-s+s\,\mathrm{Cos}[2\,th]+2\,(s+t)\,\mathrm{Sin}[th])\,\sqrt{\Big(\mathrm{Sec}[th]^2\,\big(l1^2+2\,l2^2+2\,l2\,s+2\,s^2+2\,s\,t+t^2+\big(l1^2-2\,l2\,s-2\,s^2-2\,s\,t-t^2\big)\,\mathrm{Cos}[2\,th]-(4\,l2+3\,s)\,(s+t)\,\mathrm{Sin}[th]+s^2\,\mathrm{Sin}[3\,th]+s\,t\,\mathrm{Sin}[3\,th]\big)\Big)}\Big)\Big/\Big(l1^2+l2^2\,\mathrm{Sec}[th]^2-2\,l2\,(s+t)\,\mathrm{Sec}[th]\,\mathrm{Tan}[th]+\big(2\,l2\,s+2\,s^2+2\,s\,t+t^2-2\,s^2\,\mathrm{Sin}[th]-2\,s\,t\,\mathrm{Sin}[th]\big)\,\mathrm{Tan}[th]^2\Big),\ 0,\ 0\Big\}$$

for
$\theta < \theta_E$.

Fig. 13 shows the results for the torque/tension on the beam as a function of θ for the nominal case.

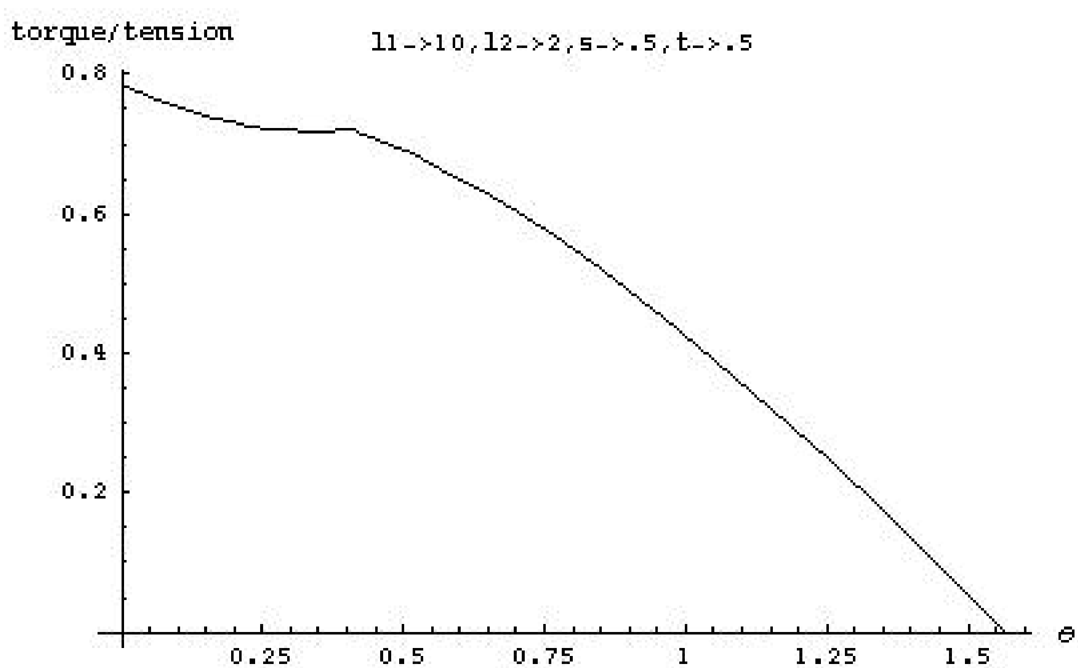

Fig. 13. The Torque over the tension as a function of beam angle.

In this graph, θ = 0 corresponds to the beam being horizontal and π/2 to it being vertical. The discontinuity in the slope of the torque corresponds to where the strand "runs into" the point of attachment of the ends of the strand.

During the change in angle, of course, the tension is not a constant, so to get the torque as a function of angle we need to multiply the torque over the tension by the tension. The tension is proportional to the fractional change in length of a strand. This is given for one of the half-strands as

L(θ)/L = Sqrt[(pf(th)-af).(pf(th)-af)]/Sqrt[(pf(π/2)-af).(pf(π/2)-af)]

which looks like

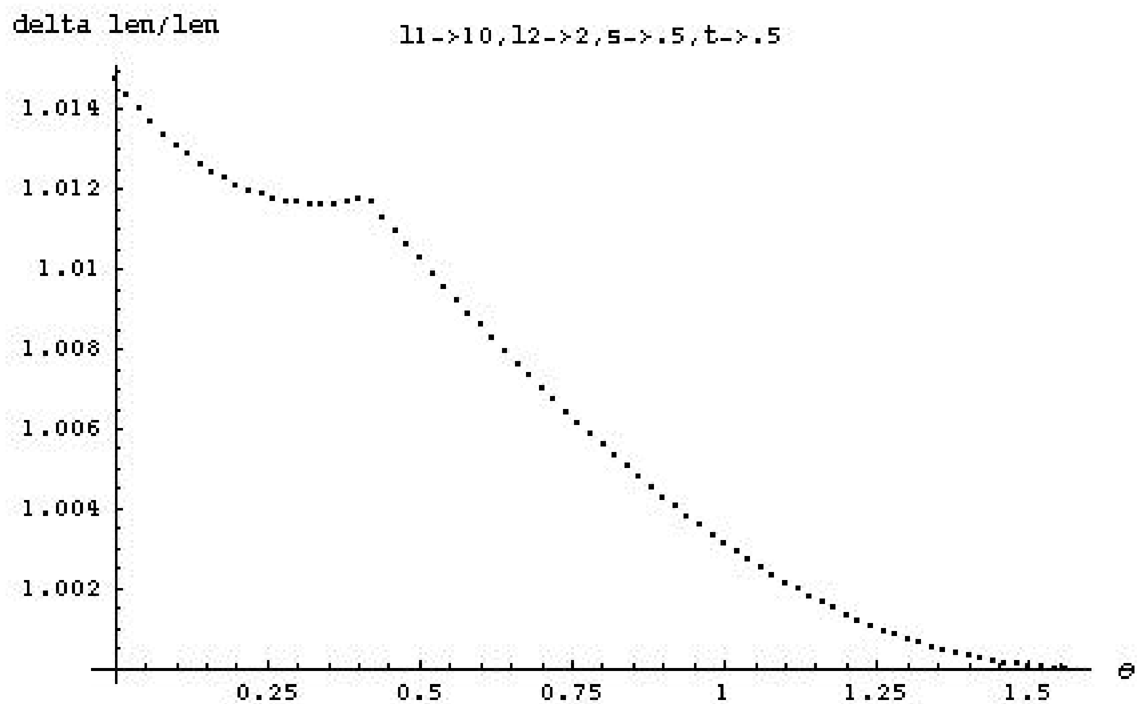

Fig. 14. The fractional change in length for a half-strand of the 2 stranded Onager.

The length changes by about 1.4%. The torque divided by the elastic constant then looks like

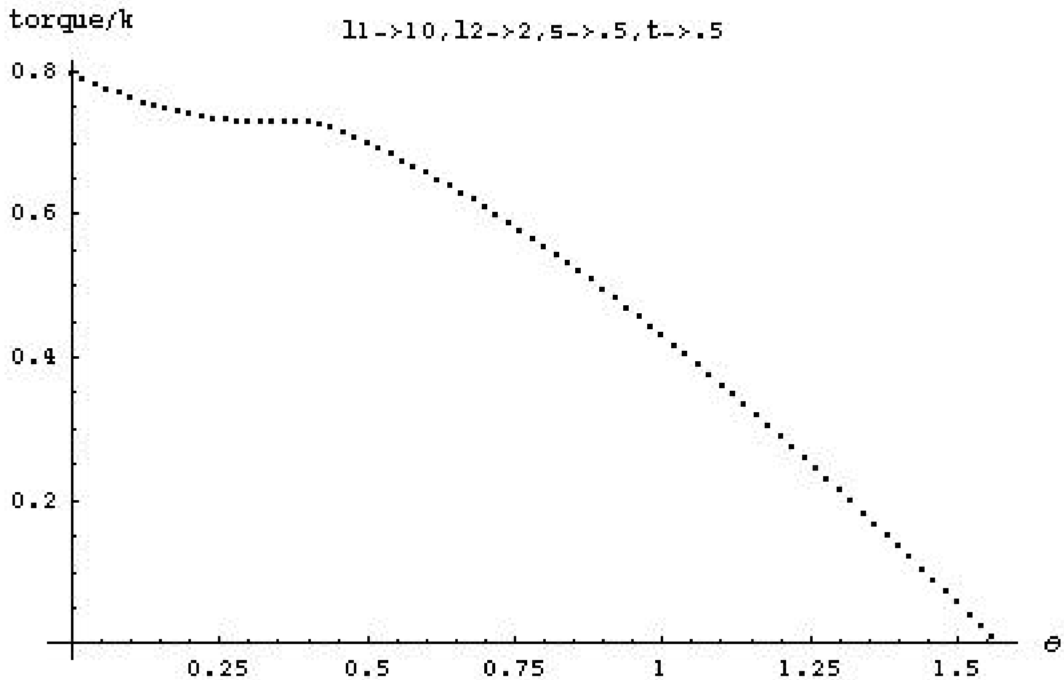

Fig. 15. The torque divided by the elastic constant of the rope for the 2 stranded Onager.

This is very similar in shape to the torque/tension plot for the nominal case.

The work involved in cocking the Onager by moving the beam from $\theta=\pi/2$ to $\theta=\theta min$ is given by the area under the curve between these limits.

The torque as a function of angle for multistranded parallel bundle models can be derived by similar methods.

Four Stranded Parallel Bundle Model

The next most complicated case for the parallel bundle model would have four parallel strands. This introduces one qualitatively new element--the friction between the strands. We will defer the discussion of this for now.

Then the eight half-strands at the points of attachment has coordinates, using the notation from before, of

Auf=(-l1,t+s,s)
Alf=(-l1,t+s,-s)
Aub=(-l1,-t-s,s)
Alb=(-l1,-t-s,-s)

and

Buf=(l1,t+s,s)
Blf=(l1,t+s,-s)
Bub=(l1,-t-s,s)

$$B_{lb}=(l1,-t-s,-s).$$

These are assumed not to move during the cocking or throw. The yoke is assumed to be rigid and the friction between the points of attachment and the strands are very high. And the friction between adjacent strands will be assumed to be zero for now.

The points of contact of each strand with the beam when it is oriented along the y axis (i.e., beam is cocked) can be seen to be

$$P_{uf}=(0,t+s,l2)$$
$$P_{lf}=(0,t+3s,l2)$$
$$P_{ub}=(0,-t-3s,-l2)$$
$$P_{lb}=(0,-t-s,-l2),$$

and the vectors from the origin to the point of application of the forces on the beam are

$$r_{uf}=P_{uf}$$
$$r_{lf}=P_{lf}$$
$$r_{ub}=P_{ub}$$
$$r_{lv}=P_{lb}$$

The unit vector for the tension in each strand goes from the beam to the point of attachment. There are eight half-strands: four go to the left and four to the right from the points of attachment. Use L and R to denote these directions.

Front half-strands:

$$T_{uf,L} = T\,(P_{uf}-A_{uf})/|P_{uf}-A_{uf}|$$
$$T_{lf,L} = T\,(P_{lf}-A_{lf})/|P_{lf}-A_{lf}|$$
$$T_{uf,R} = T\,(P_{uf}-B_{uf})/|P_{uf}-B_{uf}|$$
$$T_{lf,R} = T\,(P_{lf}-B_{lf})/|P_{lf}-B_{lf}|$$

and similarly for the back half-strands.

As in the two stranded parallel bundle model, the upper front strand is hindered in its movement by the width of the attachment, and since the lower strand is parallel to and touching the upper strand, its movement will be hindered also.

We show in the Appendix III, using the Mathematica programming language the calculations of each of the torques about the origin and their sum. It is not hard to do this by hand, but Mathematica is quicker and more reliable.

As expected, the torque lies along the x-axis, as expected by symmetry considerations. The total torque divided by the tension in one of the strands (assumed for now to be the same in both) when the beam is cocked is now a little more complicated:

$$\frac{4\,(l2 - s)\,(s + t)}{\sqrt{l1^2 + (l2 - s)^2}} + \frac{4\,(l2\,(s + t) + s\,(3\,s + t))}{\sqrt{l1^2 + 4\,s^2 + (l2 + s)^2}}$$

where 2 l2 is the height of the beam, s is the radius of the strand, and 2t is the thickness of the points of attachment. In the limiting case when s=t=0, the torque is also zero, as expected. This verifies our notion that the essential element of the model is in the finite diameter of the ropes.

The first term in the torque comes from the uf and lb strands, and the second from the lf, ub strands.

A comparison can be made between this two stranded and the four stranded models using these formulas.

For the limit when t is zero, l1=10, l2=2, say, and the s is small, the ratio is seen to be 2., as expected because the strand numbers is in the same ratio. When, say l1=10, l2=2, t=.5 and s=.5, then the ratio of the two torques is 2.23. This means that the torques are not simply proportional to the number of strands--it is a function of rope radius too.

The angular dependence of the torque/tension for this model is qualitatively similar to that for the 2 stranded parallel bundle model--it has two regions that are continuous, etc.

The General Parallel Bundle Model as a Function of Angle

It is possible to carry these calculations further into a general treatment for the n-stranded parallel bundle model as a function of angle. This is done in Appendix IV, where the calculations are illustrated for the 4 stranded parallel bundle. This set of calculations could also serve as a starting point for a detailed consideration of more complex configurations of the skein, in which, for example, there are layers of parallel strands. This has not been done yet.

One point to make about the parallel bundle model is that the tension in each strand may differ because each of the strands stretch by different amounts. Thus, there are two possibilities to consider.

In the first, which we may denote as the "fixed ends" parallel bundle model, the attachment points of each strand is assumed not to move during the cocking. It would be as if each strand were individually fastened to its point of attachment. The tension in each strand could differ, according to how much tension is initially applied and how much each is stretched during the movement of the beam.

In the second model, which may be called the "free ends" model, each of the strands move freely through the points of attachment during the movement of the beam, and the tension in all of the strands are the same.

Because the hard part of the calculation for the torque actually only calculates the torque/tension for each strand, it is easy to calculate the total torque for either of the two models, and this is also illustrated by way of an example in Appendix IV.

At this point in our investigation of the Onager, it is not possible to say when either of the models apply. Possibly the truth is somewhere between these two extremes, and it would be best to perform both calculations and average the results in some way.

One important consideration in connection with the two models is that they give different predictions on whether or not the rope will break or permanently deform during the rotation of the beam. When s and t are relatively large, the length changes are greater, and may exceed the maximum strain that the rope can sustain without permanent deformation or breaking. In the design of any Onager, it is a fairly simple matter to calculate the length changes for either the fixed or free end models, and compare them to the allowed length changes for the material of choice. Nothing is more frustrating than to make an Onager, string the strands, and have it break during cocking!

General Rules for Parallel Bundle Models

One important observation to note in all of these models, is that when the distance between the attachment points (2 x l1) increases, the torque decreases, because l1 appears only in the denominators. Also, with increasing l1, the force tending to break the yoke increases too. Other things being equal, it is better to keep the yoke breadth small for a good design.

For a given tension, rope with a big diameter (2 x s) will give better results. A thick (2 x t) point of attachment gives better results, but note that the analysis implicitly assumed that s+t <l 2: the rope is in contact with the beam during the rotation of the beam. Finally, for any l1, the torque is greater for thicker (2 x l2) beams. For small s, the torque is proportional to the number of strands, but rises faster than linearly for non-zero s and/or t.

Tensioning the Ropes and Friction at Points of Attachment

The methods available to apply the tension in the ropes during the stringing of the bundle and the problems associated with this are topics which are important for any practicing Onager engineer. A lot of work is required to accomplish it, and it is easy to go wrong. It will be clear from our discussion that much remains to be discovered.

Consider first the parallel bundle model. Normally, one does not use individual ropes tensioned between the points of attachment at the yoke, but rather one continuous rope that goes over a cylindrical bar of metal in the capstan(s), snaking between these supports a certain number of times, which is then tied off somehow at one end. There are lots of ways to actually generate this configuration.

Usually, the supporting fixture where the ropes make their turn at the points of attachment have a straight cylindrical cross-section, though we should, perhaps, keep an open mind about this.

Consider the generalized configuration in Fig. 16, in which a rope passes over

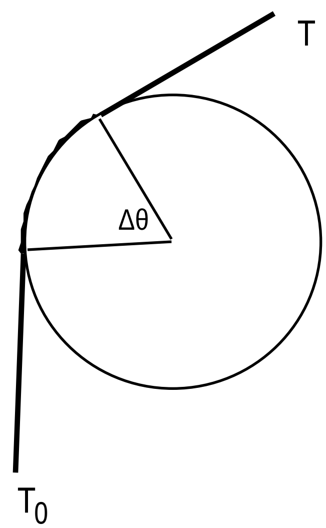

Fig. 16. A rope contacting a rough cylinder through an angle $\Delta\theta$. The tensions T and T_0 in the two segments of the rope are different.

a rough cylindrical surface, representing the supporting fixture. The cylinder is fixed so it cannot move, and the rope is held in static equilibrium by two tensions, T and T_0. The rope is in continuous contact with the supporting fixture between the points P1 and P2, which subtends an angle at the axis of the cylinder through an angle $\Delta\theta$. The static coefficient of friction between the rope and the supporting fixture is taken to be μ.

We all know from experience that if we make a couple of wraps around a pole, we can support a much greater load than without it. This is how one "brakes" moving heavy loads. Evidently the two tensions T and T_0 are not the same. In fact, it can be shown that the relationship between the two is

$$T = T_0 \, e^{\mu\Delta\theta}$$

This shows that the relationship is exponential in the friction coefficient and the subtended angle. For example, if we take just two turns around the cylinder, $\Delta\theta$ would be 4π, and if the coefficient of friction is, say, 0.5, then the ratio of the two tensions is $e^{2\pi} = 535$. This is pretty big!

For the case of the Onager, each turn in the rope would have $\Delta\theta = \pi$. Passing the rope through two of these turns will give a ratio in the tensions, then, of

$$T/T_0 = e^{\mu\pi} e^{\mu\pi} = e^{2\mu\pi}$$

In the case of passing the rope through n turns would give a ratio of the initial and final strands of rope,

$$T/T_0 = e^{\mu n \pi}.$$

So what happens when you just try to thread all of the ropes over their supporting fixtures a few times, then pulling on the rope in preparation to tying off the end? Well, the first strand in your hand would have a tension, T_0, the next one $T_0\, e^{-\mu \pi}$ and the next one would have a tension $T_0\, e^{-\mu 2 \pi}$ and so on. Suppose the friction coefficient is 0.5, so $e^{\mu\pi} = 4.8$. Then after only 3 turns, giving 4 strands, the tension in the farthest strand is only about 1% of the tension of the one in your hand! This would mean, of course, a great decrease in the efficiency of the bundle. The first strands would contain very little energy.

One way to avoid the problem would be to make the friction between the rope and the supporting fixture vanish by using a bearing of some kind. Another possibility would be to somehow clamp each rope to the supporting fixture after it is threaded and tensioned then thread, tension and clamp the next strand, and so on. A third alternative is to use individual ropes in each strand, tensioning and tying them off separately. This would have the advantage that ropes of different elasticity and diameters might be used, tailoring their properties to their relative position in the bundle. Exact ly how this problem was solved in the historical Onager is unknown.

In the twisted model bundle, this problem is largely avoided because the rope is threaded through the attachments and then the tension is applied to all of them as a group by twisting the capstans. Each strand will initially have different, relatively low, tensions at the start of the twisting. If the bundle is highly twisted, each strand will probably have more or less the same tension. The details of this situation are, however, complicated when considered in detail.

Clearly, our treatment of this leaves some room for improvement.

Internal friction in the Bundle

As we have seen, during the movement of the beam, the strands change lengths by different amounts. This means that there will be some friction between adjacent strands, provided there is a force tending to draw them together. The situation will be different for the different models.

In the parallel bundle model, we have assumed that the points of contact of the strands with the beam do not move. Friction is very high there, the same as if the strands are fastened to the beam by some sort of fastener. The model also assumes that the point of attachments are fixed and the strands are not twisted over one another.

A critical assumption in this model, which is worth reiterating, is that the vector, T, representing the tension in a strand goes from the center of the strand at the attachment point to the point of contact of the strand with the beam. The line of contact between adjacent strands in the bundle is different, however. The line of contact between the strands terminates at the point of contact of the next higher strand, but it starts at a point displaced in the z-direction by s, the radius of the strand. Call this (unit) vector line of contact, between these two points, R. These vectors are different, so the normal force between the strands at the line of contact is the cross product T x R . The friction between the two strands is proportional to this normal force, with the constant of proportionality being the coefficient of friction.

The friction between the two strands is then μ T x R, where μ is the coefficient of (kinetic) friction of the material in the strands. The net tension in a strand that is used to create the torque is then

$$T_{eff} = T - \mu\ T \times R/2,$$

where the factor of two enters because two strands are involved.

If the strands changed lengths by the same amounts during the movement of the beam, there would be no relative movement between adjacent points on the strands and so no work would be expended. If the difference in length of the two strands changed by an amount dL, the amount of work would expended would be

$$dW = (\mu\ T \times R) \cdot dL,$$

Integrating over the length of each strand gives, finally, the work expended in friction between the two strands i, and i+1 during the cocking as

$$W_{i,i+1} = (\mu\ T \times R) \cdot (\Delta L_{i+1} - \Delta L_i).$$

where ΔL_i is the length change in the ith strand during cocking. To get the total energy lost, we need to sum this over all of the lines of contact between adjacent strands. If we have N strands, there are N-1 of them.

During the throw, energy is also lost to friction, and the amount lost depends upon the position (angle, θ) of the beam. For obtaining the dynamics of the beam, T_{eff} summed over all of the strands, should be used, rather than T.

Onager efficiency

A definition of the range efficiency of the onager may be easily derived. During multiple throws of the Onager, the beam is cocked by being brought from a roughly vertical position to a roughly horizontal one. The torque on the beam about its axis due to the twisting of the skein varies with the angle and is rather easily measured or (approximately) calculated. If the torque, τ as a function of the rotation angle, θ, is known, then the work, E, expended on the bundle is given by

$$E = \int_{\theta_i}^{\theta_f} \tau(\theta)d\theta$$

During the throw, this energy is partly transferred into the projectile of mass m_2, and partly into the rotation of the beam of mass mb, and into friction, and other imperfections. If none of the energy went into the beam and none into heat, and the projectile left the sling (at 45° with respect to the horizontal),for maximum distance, it would go a distance

$$Rm = 2\ E/(m_2\ g).$$

Any real Onager will throw the projectile a distance, Rexp, different from this because of various sources of friction.

The measured range efficiency for a real Onager would then be

$$Reff = Rexp/Rm.$$

This has apparently never been measured for any actual Onager.

A theoretical maximum efficiency for a design would involve calculating the integral exactly to get Rm, and then carrying out a simulation of the dynamics of the throw, using a non-zero value for the mass of the beam to get a theoretical maximum range of Rth. The range efficiency for the design would then be

$$Reff = Rth/Rm.$$

Energy in the Bundle

The energy in the bundle available for rotating the beam is a key parameter in designing the Onager. This can and should be understood and calculated for your design. This requires only some pretty elementary considerations of the geometry and material used to make the bundle.

First, whatever material is used, we must understand that there is a maximum strain (fractional length change) that it can withstand without permanently deforming or breaking. Our design should not exceed this maximum.

At this maximum strain, the strength of a rope is called its maximum working strength. The strength of a rope is proportional to the cross-sectional area of the rope.

If the rope is elastic and below the elastic limit, the energy, E, per unit volume, V, of rope contained in a rope with a strain of dl/l and a stress σ is

$$E/V = (dl/l) * \sigma/2$$

Here stress is measured in N/m^2 or Pa, in the metric system, and psi in the American system. The strain is dimensionless because it is a fractional length change.

In order to look at the trade-offs involved in selecting a material for the rope in the bundle, one has to survey the properties of ropes made from various materials, with different braids, from different manufacturers, and so on. Our initial attempt at this exercise looked at some specifications of different ropes from two websites (4,5). I looked for strengths of rope that were 1/2 inch in diameter, and converted that to a stress (strength/area). I then calculated the energy in
one meter of this rope when stretched to the maximum (tabulated) strain, using the formula above. The table shows the results.

material	strain at max stress	strength of 1/2" dia	str./area	E for 1 m long at max strain	E for 1 m at 2% str.
		lbs	N/cm^2	MJ	MJ
nylon double braid	0.32	8500	7500	0.61	0.038
nylon 8 stranded	0.32	6500	5700	0.46	0.029
polyester double braid	0.17	8200	7200	0.31	0.036
polypropylene 12 strand	0.2	4900	4300	0.22	0.022
manila	0.11	3300	2900	0.08	0.015
steel	0.02	2000	1760	0.01	0.010
kevlar K	0.01	15000	13200	0.03	breaks

Clearly, a well-designed bundle has the maximum energy per unit volume of rope. The strain in the rope has to be below the maximum, but as high as possible. Looked at in this way (respective of cost) nylon double braid appears to be the best candidate and is definitely superior to polypropylene. Steel cable is not even in the running. Kevlar, though very strong, is also ruled out because its maximum strain is so low. The table can be used to quickly estimate the values for other sizes of ropes: the energy E is proportional to the length and directly proportional to the square of the diameter of the rope.

For a parallel bundle model, we can use the energy formula in a very straightforward way. If the distance between attachments is 2 l1 and there are n strands, each with a diameter 2 s, then we have a volume of rope (neglecting the small length at the attachment), for the parallel bundle model, given approximately by

$$V = 4\ \mathrm{sqrt}(l1^2 + l2^2)\ n\ \pi\ s^2$$

so the energy in the bundle, using the free ends, parallel bundle model is

$$E = 2\ n\ \pi\ s^2\ (dl/l)\ \mathrm{sqrt}(l1^2 + l2^2)\ \sigma.$$

As an example, take a four stranded parallel bundle with s = 0.5 cm, l1= 5 cm, l2 = 1 cm, made from nylon with a stress at failure of 57 MPa, and a strain of 0.32 it would have an energy of

$$E = 8\,\pi \times 0.005^2 \times 0.32 \times \mathrm{sqrt}(0.05^2 + 0.01^2) \times 57 \times 10^6 =$$

$$648\ \mathrm{J}$$

This assumes all of the strands are strained almost to breaking, equally.

The range of projectile with a velocity v at 45° with respect to the horizontal is v^2/g, where g is the acceleration due to gravity. If all of the energy E goes into kinetic energy of a projectile having a mass m, therefore, the range will be

$$R = 2\,E/(m\,g).$$

Therefore if all of the energy is deposited into a projectile (i. e., efficiency = 100%, no friction) having a mass, say, of 1000 g, that fires off at 45°, it will go

$$R = 2 \times 648/(1 \times 9.8) = 132\ \mathrm{m},$$

which is quite respectable. However, as we have showed in the Appendix IV for this model (having the same l1, l2 , s and t) the strains when cocked in each of the half-strands varied from 0.014 up to 0.27 with an average of about 0.10. This alone decreases the range by a factor of about 0.1/0.32 to 41 m.

The coupling efficiency between the bundle and the projectile will be something less than 100%, mostly because the bundle must accelerate the beam to some high rotational velocity when the projectile leaves the sling, which leaves some time before the beam is upright. Inefficiency due to this might be guessed to be (a simulator can provide this number) on the order of 0.5, so we might expect a range for this Onager on the order of 20 m. The overall efficiency of this design would then be on the order of 0.3*0.5 or 15%.

If the overall efficiency of the Onager is ε, then the range will be

$$R = (2\,\varepsilon\,n\,\pi\,s^2\,(dl/l)\,l1\,\sigma)/(m\,g).$$

This formula can be used for measuring the efficiency of a real Onager: measure the range and solve for the efficiency. This has not been done so far, apparently, by any Onager engineer.

Back of the Envelope Design

In the parallel bundle models, the torque is zero when $\theta = 90°$, and some maximum torque t_c when cocked.

If the torque is linear with theta, then the integral can be approximated as the area of a triangle, so

$$E = \pi\, \tau_c / 4.$$

Combining these relations shows that an estimate for the range would be

$$R_{est} \approx \pi\, \tau_c / (2\ m_2\ g),$$

where g is the acceleration due to gravity.

This formula can serve as a starting point for a first approximation for a design. As an example, suppose you want to throw a 2 kg pumpkin 100 m, then you need a design that can produce a torque of about

$$\tau_c \approx (2/\pi)\ m_2\ g\ R_{est} = 2\ \text{kg x } 9.8\ \text{m/s}^2 \text{ x } 100\ \text{m}$$

$$\tau_c \approx 2000\ \text{kg m}^2\ \text{s}^{-2} = 2000\ \text{N m}.$$

Allowing for various inefficiencies in the design, one might want, say, twice this much, or 4000 N m. Try the very simplest model for a start: the two stranded parallel bundle model, which has

$$\tau_c = \frac{4\ l2(s+t)}{\sqrt{l1^2 + l2^2}}\ T$$

Starting with thickness of the attachment 2t = 0, and with attachment point separation of, say 2 l1 = 50 cm and beam thickness 2*l2 = 10 cm gives a required

$$s\ T = \frac{\sqrt{l1^2 + l2^2}}{4\ l2}\ \tau_c = \frac{\sqrt{0.25^2 + .025^2}}{4\ 0.05}\ \tau_c = 1.3\ \tau_c \approx 5200\ \text{N m}$$

So if the rope has a radius of s = 1 cm, then we'd better have one that can sustain a tension of 520,000 N without breaking. A useful conversion factor for the Americans is that 1 N is 0.2248 pounds of force. So we'd need a rope that can hold about 120,000 pounds. This would be an awful strong rope! Better assume a bigger rope, with more strands.

Suppose our nylon rope 0.5 in in diameter has a working strength of about 4000 pounds (it is a little old), so s T for the rope is about 0.5 x 2.54 cm/in x 4000 p/0.2248 = 225 N m which is still off from the required 5200 Nm by a factor of 23.

Let's try a model with t=s, and 2 l1 decreased to 30 cm. Then s T would be required to be about 0.4 τ_C = 1600 N m, which is better by a factor of 3.25, and 8 strands of nylon rope 0.5 in in diameter would about do it.

Conclusions

Two basic models for the rope bundle have been described. The parallel bundle model, and highly twisted model with two variants, "open" and "closed" are described. The geometry involved in the models are described in terms of the dimensions of the parts, and expressions for the torque and tension in a strand as a function of angle are worked out.

By using some elementary physical reasoning, we have been able to show where some of the guidelines for a good Onager design comes from: the key is to consider the elementary strand in the bundle as having a non-zero radius. The principle resulting guideline is to make the yoke braces close together, the beam, points of attachment, and the rope should all be thick. Some simple dimensional analysis showed that the energy in the rope scales as the third power of its diameter, and the thickness of the beam should be scaled as the first power of the skein (rope) diameter.

Because there is a limit in the travel of the strands in the parallel bundle model during cocking, the torque-cocking angle graph has two segments to it, which would need to be incorporated in any accurate dynamical simulations of this class of models.

A method for measuring the experimental range efficiency for a real Onager has been obtained, and a method to calculate a theoretical range efficiency for simulated Onagers has also been described. A method for starting a back-of-the envelope design of the bundle based on bundle energetics was illustrated by an example. It appears that nylon is probably the best available material for the rope.

Much work remains to be done to improve our understanding of Onager design, but a good start has been made.

References

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2. Payne-Gallway, Sir Ralph, The Crossbow, The Holland Press, London, England, 1958.

3. Siano, D. B., Will It Break? , http://members.home.net/dimona/ , 2000.

4. http://www.usrigging.thomasregister.com

5. http://www.wallrope.com

Appendix I

The Torque

The magnitude of the torque of a force about a given axis of rotation is defined as the product of the magnitude of the component of the force in a plane perpendicular to the axis by the lever arm of the force relative to the given axis. The lever arm is the perpendicular distance from the axis of rotation to the line of action of the force. This will be clear by considering the diagram

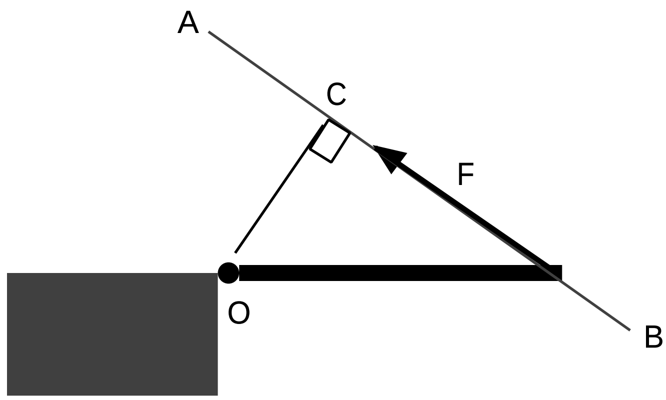


Fig. A1. The definition of torque, using a door fastened by a hinge at O.

The diagram shows a door fastened to a fixed wall by a hinge at O that allows rotation about an axis through O perpendicular to the plane of the paper. We apply a force, F, to the other end of the door at some angle with respect to the door. The line of action of the force is then A-B, and the lever arm is the length of O-C, which we can call L. The magnitude of the torque is then L times F. It has dimensions of length times force, N m in the metric system, or ft p in the English system.

Note that if the Force were perpendicular to the door, the torque would be maximal, because the line of action of the force is as big as it can get. And if F were along the door, the line of action, and hence the force, would be zero. Note that we have only considered a force F that is in the plane of the paper. If the force were applied at an angle to the plane of the paper, we would only consider the component of the force in the plane of the paper.

So far we've only discussed the magnitude of the torque. But torque is a vector. It's direction in the case shown is perpendicular to the paper, coming out of the paper.

A little more sophisticated definition of torque is shown in Fig. 2 which represents a force, F, being applied to a body at a point of application, A. Think of a string attached at A being pulled with a force F.

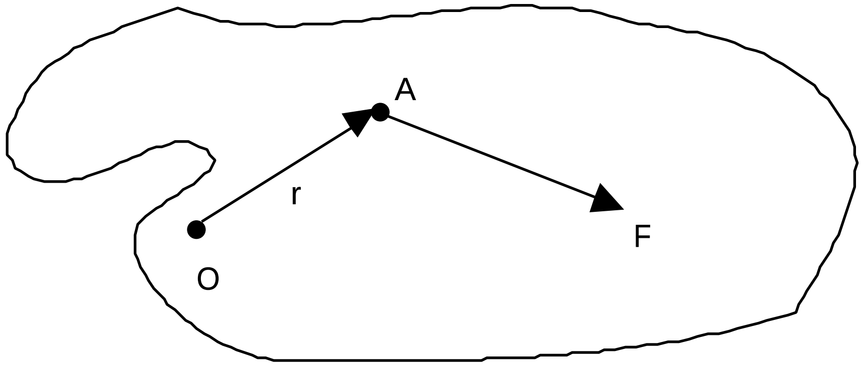


Fig. A2. The general definition of torque about a point O produced on a solid body by a force.

Point O is some point that we want to consider the rotation of the body about. A is at a distance r from that point. Note that the vectors r and F are not necessarily in the plane of the paper. Then the torque, τ, about the point O is given by the cross product of the two vectors r and F. In conventional vector notation,

$$\tau \equiv r \times F.$$

It should be clear that the point 0 and the vector f define a plane because a line and a point exterior to the line define a plane. Therefore r and F are in that plane, and we can define an angle, ϕ between them. The magnitude of the torque is then $\tau = r F \sin(\phi)$. So that when r and F are perpendicular, the torque is maximal ($\sin(\phi) = 1$) and when they are parallel, the torque is zero. The vector sense of the torque is perpendicular to the plane defined by O and F, and is given by the right hand rule.

Further, if we define an xyz right handed coordinate system with unit vectors i, j and k, r has components r1, r2 and r3. And similarly for F. Then the definition of the cross product for these two vectors is given as

$$\tau = i(r2\ F3 - r3\ F2) - j(r1\ F3 - r3\ F1) + k(r1\ F2 - r2\ F1).$$

Thus if we know all of the components of the force, the point of application of the force, and the vector components of the radius from the axis of rotation of the beam, we can easily calculate the torque.

The Couple

There is a special configuration of forces that is useful to understand. This is a case where two parallel forces of equal magnitude but opposite direction are applied at separate points in a body. Since the lines of action of the forces are parallel, they lie in a common plane. The perpendicular distance between the two forces is called the moment arm of the couple. The magnitude of the couple is F d. Obviously, it produces a tendency to turn the body, just like a torque, and it has the same dimensions as a torque.

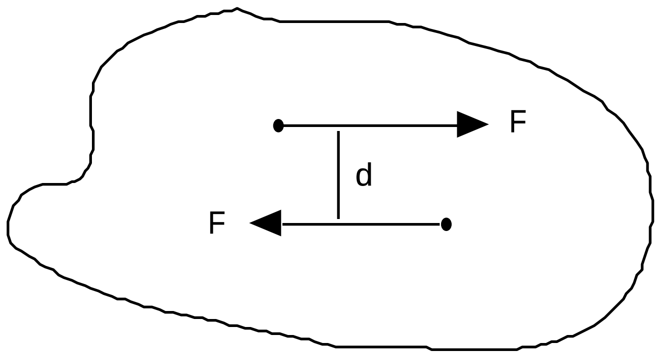


Fig. A3. Definition of a couple from two opposite and equal forces in a plane

Appendix II

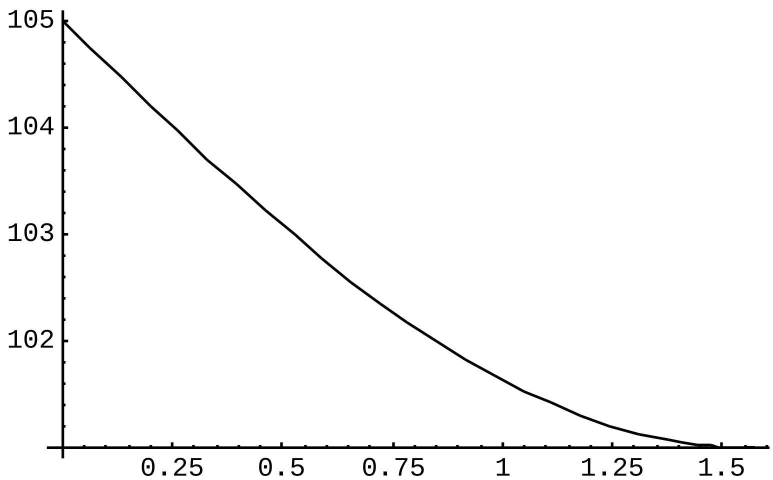
```
(* 2 stranded parallel bundle Onager*)
(* Donald Siano*)
(* May 6 2001  *)
(* get torque on Onager beam for general angle of beam
with horizontal, theta*)

(* u means upper
        l means lower
        f means forward
        b means back
        a is attachment at left
        b is attachment at right
        p is point of contact of strand

        2t is thickness of attachment
        2s is diameter of single strand*)

        (* 2 strands attached at left:*)
af={-l1,t+s,0};
ab={-l1,-t-s,0};


        (* 2 strands attached at right:*)
bf={l1,t+s,0};
bb={l1,-t-s,0};(*first study cocked case, then the upright
case, then general case:*)

(* points of contact with beam, cocked;*)

pf={0,s+t,l2};
pb={0,-s-t,-l2};

(* unit vectors in direction of tension in indicated half-
strand*)
TfL=Simplify[( -pf+af)/Sqrt[(pf-af).(pf-af)]]
TfR=Simplify[(- pf+bf)/Sqrt[(pf-bf).(pf-bf)]]
```

```
TbL=Simplify[(- pb+ab)/Sqrt[(pb-ab).(pb-ab)]]
TbR=Simplify[(- pb+bb)/Sqrt[(pb-bb).(pb-bb)]]

TfL+TfR+TbL+TbR (*check*)

{-(l1/Sqrt[l1^2 + l2^2]), 0, -(l2/Sqrt[l1^2 + l2^2])}
 {l1/Sqrt[l1^2 + l2^2], 0, -(l2/Sqrt[l1^2 + l2^2])}
 {-(l1/Sqrt[l1^2 + l2^2]), 0, l2/Sqrt[l1^2 + l2^2]}
 {l1/Sqrt[l1^2 + l2^2], 0, l2/Sqrt[l1^2 + l2^2]}
{0, 0, 0}

(*look at signs for
   TfL:  force on beam at pf from the front strand is to left
and down,
  which are correct and sum of forces on the beam when
cocked is zero*)
(* the radius from origin to point of action of tensions*)
rf=pf;
rb=pb;

(* resultant for the dimensionless tension of the two front
half-strands*)
tf=TfL+TfR
(*  and two back strands:*)
tb=TbL+TbR


{0, 0, -((2*l2)/Sqrt[l1^2 + l2^2])}
{0, 0, (2*l2)/Sqrt[l1^2 + l2^2]}

(* this shows that the front strand pushes only down,
  and the back strand pushes only up, which is correct*)
(* torque on the beam when cocked:*)
tk1=Simplify[CrossProduct[rf,tf]+CrossProduct[rb,tb]]

{-((4*l2*(s + t))/Sqrt[l1^2 + l2^2]), 0, 0}

(* this is in the -x-direction, which is correct*)
TfL
TfR
```

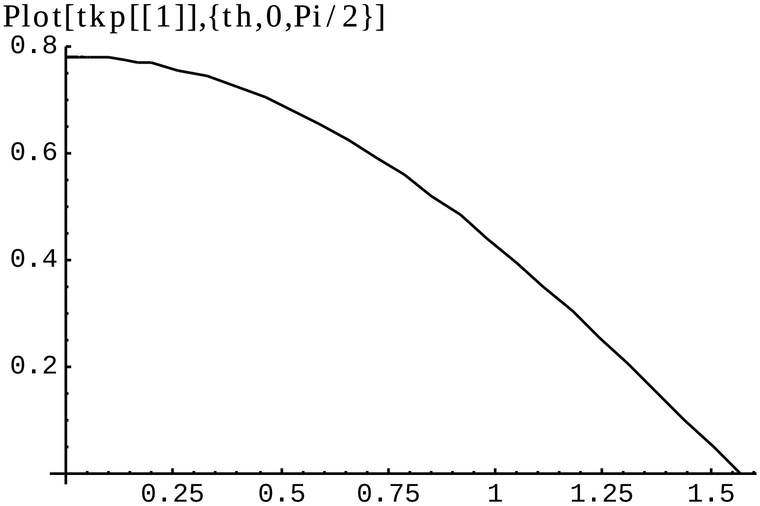
```
{-(l1/Sqrt[l1^2 + l2^2]), 0, -(l2/Sqrt[l1^2 + l2^2])}
{l1/Sqrt[l1^2 + l2^2], 0, -(l2/Sqrt[l1^2 + l2^2])}

TfL+TfR+TbL+TbR

{0, 0, 0}

(*for the upright beam,*)
(* points of contact with beam, cocked*)
pf={0,l2,0};
pb={0,-l2,0};
(* unit vectors in direction of tension in indicated half-
strand*)
TfL=Simplify[( pf-af)/Sqrt[(pf-af).(pf-af)]];
TfR=Simplify[( pf-bf)/Sqrt[(pf-bf).(pf-bf)]];
TbL=Simplify[( pb-ab)/Sqrt[(pb-ab).(pb-ab)]];
TbR=Simplify[( pb-bb)/Sqrt[(pb-bb).(pb-bb)]];


(* the radius from origin to point of action of tensions*)

rf=pf;
rb=pb;


(* resultant for the dimensionless tension of the two front
half-strands*)
tf=TfL+TfR
tb=TbL+TbR

{0, (2*(l2 - s - t))/Sqrt[l1^2 + (-l2 + s + t)^2], 0}
{0, (2*(-l2 + s + t))/Sqrt[l1^2 + (-l2 + s + t)^2], 0}

tk1=Simplify[CrossProduct[rf,tf]+CrossProduct[rb,tb]]
(* should be zero for upright beam*)

{0, 0, 0}

(*Now for the general case where beam is at angle th,*)

(* points of contact with beam, while y < t+s  *)
```

```
pf={0,l2 Sin[th],l2 Cos[th]};
pb={0,-l2 Sin[th],-l2 Cos[th]};
(* unit vectors in direction of tension in indicated half-
strand*)
TfL=Simplify[( pf-af)/Sqrt[(pf-af).(pf-af)]];
TfR=Simplify[( pf-bf)/Sqrt[(pf-bf).(pf-bf)]];
TbL=Simplify[( pb-ab)/Sqrt[(pb-ab).(pb-ab)]];
TbR=Simplify[( pb-bb)/Sqrt[(pb-bb).(pb-bb)]];


(* the radius from origin to point of action of tensions*)

rf=pf;
rb=pb;


(* resultant for the dimensionless tension of the two front
half-strands*)
tf=TfL+TfR
tb=TbL+TbR
tk1=Simplify[CrossProduct[rf,tf]+CrossProduct[rb,tb]]
(* torque is Tension times this:*)

{0, -((2*(s + t - l2*Sin[th]))/
    Sqrt[l1^2 + l2^2*Cos[th]^2 + (s + t - l2*Sin[th])^2]),
  (2*l2*Cos[th])/Sqrt[l1^2 + l2^2*Cos[th]^2 + (s + t - l2*Sin[th])^2]}

{0, (2*(s + t - l2*Sin[th]))/
   Sqrt[l1^2 + l2^2*Cos[th]^2 + (s + t - l2*Sin[th])^2],
  -((2*l2*Cos[th])/Sqrt[l1^2 + l2^2*Cos[th]^2 + (s + t - l2*Sin[th])^2])}

{(4*l2*(s + t)*Cos[th])/Sqrt[l1^2 + l2^2*Cos[th]^2 + (s + t - l2*Sin[th])^2],

  0, 0}

(* calculate length of a half-strand*)
Simplify[(af-pf).(af-pf)]

l1^2 + l2^2*Cos[th]^2 + (s + t - l2*Sin[th])^2
```

```
Plot[Simplify[(af-pf).(af-pf)]/.{l1->10,l2->2,s->.5,t-
>.5},{th,0,Pi/2}]
```

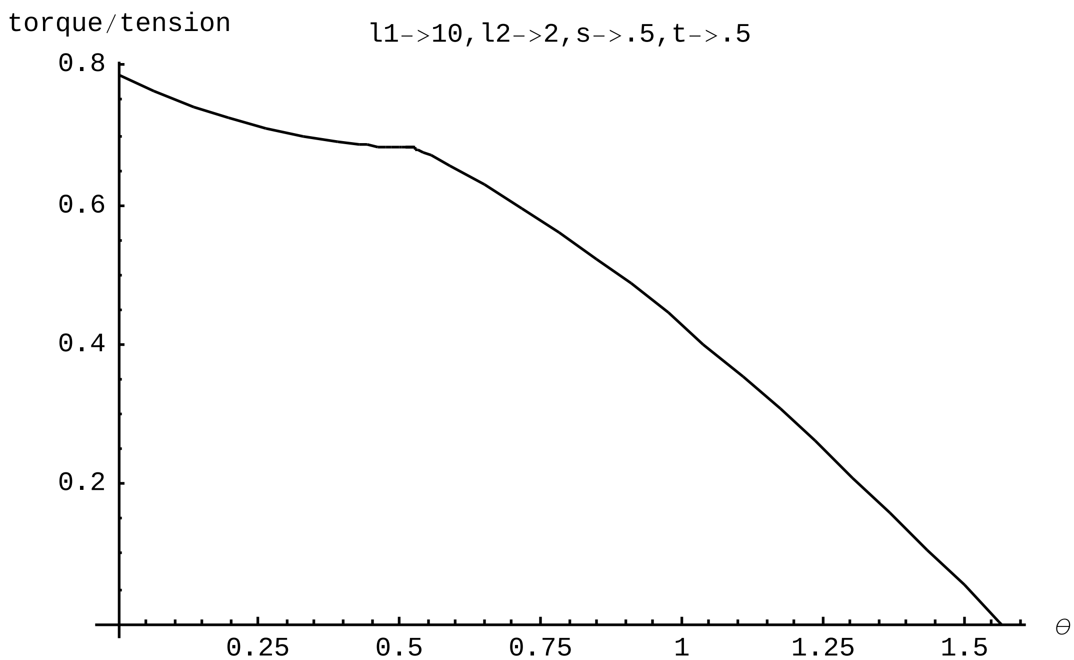



```
(*checks:*)  tk1/.th->Pi/2

{0,0,0}

tk1/.th->0
{(4*l2*(s + t))/Sqrt[l1^2 + l2^2 + (s + t)^2], 0, 0}


(* look at a typical case:*)
tkp=tk1/.{l1->10,l2->2,s->.5,t->.5}
{(8.*Cos[th])/Sqrt[100 + 4*Cos[th]^2 + (1. - 2*Sin[th])^2], 0, 0}0}

Plot[tkp[[1]],{th,0,Pi/2}]
```



```
(*for the general case where beam is at angle th,*)
(* points of contact with beam, while y<t+s*)
(*phe=ArcSin[(Sqrt[l2^2-(t+s)^2])/l2];*)
phe=.;
pf={0,t+s,-(1/Tan[phe])*(t+s-l2* Cos[phe])+l2*Sin[phe]}/.phe-
>Pi/2-th
pb={0,-t-s,(1/Tan[phe])*(t+s-l2* Cos[phe])-
l2*Sin[phe]}/.phe->Pi/2-th

(* unit vectors in direction of tension in indicated half-
strand*)
TfL=Simplify[( pf-af)/Sqrt[(pf-af).(pf-af)]];
TfR=Simplify[( pf-bf)/Sqrt[(pf-bf).(pf-bf)]];
TbL=Simplify[( pb-ab)/Sqrt[(pb-ab).(pb-ab)]];
TbR=Simplify[( pb-bb)/Sqrt[(pb-bb).(pb-bb)]];

{0, s + t, l2*Cos[th] - (s + t - l2*Sin[th])*Tan[th]}
{0, -s - t, -l2*Cos[th] + (s + t - l2*Sin[th])*Tan[th]}

pf/.th->0
pb/.th->0 (* checks ok*)
{0, s + t, l2}
{0, -s - t, -l2}

(* the radius from origin to point of action of tensions*)

rf=pf;
rb=pb;


(* resultant for the dimensionless tension of the two front
half-strands*)
tf=TfL+TfR
tb=TbL+TbR
tks=Simplify[CrossProduct[rf,tf]+CrossProduct[rb,tb]]
(* torque is Tension times this:*)
{0, 0, (2*Sec[th]*(l2 - (s + t)*Sin[th]))/
  Sqrt[l1^2 + Sec[th]^2*(l2 - (s + t)*Sin[th])^2]}

{0, 0, (2*(-l2*Sec[th] + (s + t)*Tan[th]))/
  Sqrt[l1^2 + (l2*Sec[th] - (s + t)*Tan[th])^2]}
```

```
{-((4*Sqrt[2]*(s + t)*Sec[th]*(-l2 + (s + t)*Sin[th]))/
    Sqrt[Sec[th]^2*(l1^2 + 2*l2^2 + s^2 + 2*s*t + t^2 +
        (l1^2 - (s + t)^2)*Cos[2*th] - 4*l2*(s + t)*Sin[th])]), 0, 0}

phe=ArcSin[(Sqrt[l2^2-(t+s)^2])/l2]/.{l1->10,l2->2,s->.5,t->.5}
1.0472

(* check alternate form*)
phe=ArcCos[(t+s)/l2/.{l1->10,l2->2,s->.5,t->.5}]
1.0472

the=Pi/2-phe
0.523599

(* the two sections*)
tkps=tks1/.{l1->10,l2->2,s->.5,t->.5};
tkpe=tk1[[1]]/.{l1->10,l2->2,s->.5,t->.5};

(* Now combine the two sections into one function and plot
it*)
tkgen[th_]:=If[th<the,tkps,tkpe]
Plot[tkgen[th],{th,0,Pi/2},PlotLabel->"l1->10,l2->2,s->.5,t-
>.5",
  AxesLabel->{"\[Theta]","torque/tension"}]
```




```
(*checks:*)tk1 /. th -> Pi / 2
```

{0, 0, 0}

```
tk1 /. th -> 0
```

$$\left\{\frac{4\,l2\,(s+t)}{\sqrt{l1^2+l2^2+(s+t)^2}},\ 0,\ 0\right\}$$

```
(* look at a typical case:*)
tkp = tk1 /. {l1 -> 10, l2 -> 2, s -> .5, t -> .5}
```

$$\left\{\frac{8.\,\mathrm{Cos}[th]}{\sqrt{100+4\,\mathrm{Cos}[th]^2+(1.-2\,\mathrm{Sin}[th])^2}},\ 0,\ 0\right\}$$

Appendix III

Mathematical Analysis of the 4 Stranded Parallel Bundle
when cocked

```
* 4 stranded parallel bundle Onager *)
(* Donald Siano*)
(* May 6 2001   *)


(* u means upper
 l means lower
 f means forward
 b means back
 a is attachment at left
 b is attachment at right
 p is point of contact of strand

 2 t is thickness of attachment
 2 s is diameter of single strand*)

 (* 4 strands attached at left:*)
auf = {-l1, t + s, s};
alf = {-l1, t + s, -s};
aub = {-l1, -t - s, s};
alb = {-l1, -t - s, -s};

 (* 4 strands attached at right:*)
buf = {l1, t + s, s};
blf = {l1, t + s, -s};
bub = {l1, -t - s, s};
blb = {l1, -t - s, -s};

(* points of contact with beam*)
puf = {0, t + s, l2};
plf = {0, t + 3 * s, l2};
pub = {0, -t - 3 * s, -l2};
plb = {0, -t - s, -l2};
```

```
(* unit vectors in direction of tension in indicated half-strand*)
TufL = Simplify[(puf - auf) / Sqrt[(puf - auf) . (puf - auf)]];
TlfL = Simplify[(plf - alf) / Sqrt[(plf - alf) . (plf - alf)]];
TufR = Simplify[(puf - buf) / Sqrt[(puf - buf) . (puf - buf)]];
TlfR = Simplify[(plf - blf) / Sqrt[(plf - blf) . (plf - blf)]];

TubL = Simplify[(pub - aub) / Sqrt[(pub - aub) . (pub - aub)]];
TlbL = Simplify[(plb - alb) / Sqrt[(plb - alb) . (plb - alb)]];
TubR = Simplify[(pub - bub) / Sqrt[(pub - bub) . (pub - bub)]];
TlbR = Simplify[(plb - blb) / Sqrt[(plb - blb) . (plb - blb)]];

(* the radius from origin to point of action of tensions*)
ruf = puf;
rlf = plf;
rub = pub;
rlb = plb;

(* resultant for the tension of the two front strands*)
tuf = TufL + TufR
tlf = TlfL + TlfR
```

$$\left\{0,\ 0,\ \frac{2\ (l2-s)}{\sqrt{l1^2+(l2-s)^2}}\right\}$$

$$\left\{0,\ \frac{4\ s}{\sqrt{l1^2+4\ s^2+(l2+s)^2}},\ \frac{2\ (l2+s)}{\sqrt{l1^2+4\ s^2+(l2+s)^2}}\right\}$$

```
(*resultant for the tension of the two back strands*)

tlb = TlbL + TlbR
tub = TubL + TubR
```

$$\left\{0,\ 0,\ \frac{2\ (-l2+s)}{\sqrt{l1^2+(l2-s)^2}}\right\}$$

$$\left\{0,\ -\frac{4\ s}{\sqrt{l1^2+l2^2+2\ l2\ s+5\ s^2}},\ \frac{2\ (-l2-s)}{\sqrt{l1^2+l2^2+2\ l2\ s+5\ s^2}}\right\}$$

```
(* torque from uf, lb pair*)
tk1 = Simplify[
  CrossProduct[ruf, tuf] + CrossProduct[rlb, tlb]]
(* torque from lf, ub pair*)
tk2 = Simplify[
  CrossProduct[rlf, tlf] + CrossProduct[rub, tub]]
```

$$\left\{\frac{4\ (l2-s)\ (s+t)}{\sqrt{l1^2+(l2-s)^2}},\ 0,\ 0\right\}$$

$$\left\{\frac{4\ (l2\ (s+t)+s\ (3\ s+t))}{\sqrt{l1^2+4\ s^2+(l2+s)^2}},\ 0,\ 0\right\}$$

```
(*total torque*)
tktot = tk1 + tk2
```

$$\left\{\frac{4\,(l2-s)\,(s+t)}{\sqrt{l1^2+(l2-s)^2}}+\frac{4\,(l2\,(s+t)+s\,(3\,s+t))}{\sqrt{l1^2+4\,s^2+(l2+s)^2}},\,0,\,0\right\}$$

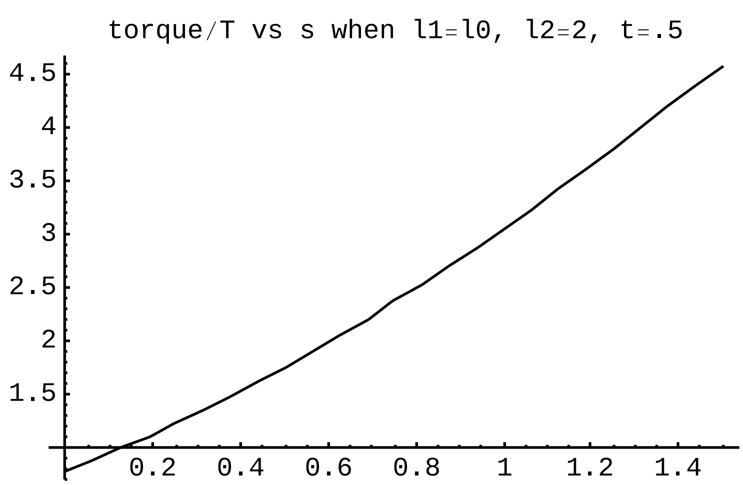
```
(* this is only in the x-direction, as expected*)

tktot /. {s -> 0, t -> 0}
(* should be {0,0,0} to agree with earlier results*)
```

{0, 0, 0}

```
(* case when t=0:*)
tktot /. t -> 0
```

$$\left\{\frac{4\,(l2-s)\,s}{\sqrt{l1^2+(l2-s)^2}}+\frac{4\,(l2\,s+3\,s^2)}{\sqrt{l1^2+4\,s^2+(l2+s)^2}},\,0,\,0\right\}$$

```
(* suppose  l1=l0,  l2=5  t=.5  and  s varies  up to  10*)
(* this  is  a  plot  of  torque /(T)  vs  s*)
tk10 = tktot [[1]] /. {l1 -> 10, l2 -> 2, t -> .5};
 Plot [tk10 , {s, 0, 1.5 },
 PlotLabel  -> "torque /T  vs  s  when  l1=l0,  l2=2,  t=.5"]
```



- Graphics -

When the beam is upright, the points of attachment are unchanged, while the point of contact with the beam becomes

```
(* points of contact with beam*)
(* beam is upright*)
puf = {0, l2, s};
plf = {0, l2, -s};
pub = {0, l2, s};
plb = {0, l2, -s};
```

The same calculations shown above can be applied again, and we easily find in this case that the total torque on the beam is zero when the beam is in the vertical direction.

Appendix IV

```
(* nparallelgenangle3.nb*)
(* calculates torque/tension, etc. for an n
  stranded parallel bundle model for an Onager
as a function of position of the beam*)

(* Donald Siano*)
(* June 14 2001  *)

(*       Notation:       *)
(* u means upper
 l means lower
 f means forward
 b means back
 a is attachment at left
 b is attachment at right
 p is point of contact of strand

 2 t is thickness of attachment
 2 s is diameter of single strand
 2 l1 is the thickness of the beam

 big means big theta, nearly upright beam
 *)
```

```
(* origin is at center of beam, which is held by
   strands between anchor points along x-axis;
 z axis is along the long axis of the beam;
 y axis is perpendicular to these.*)

(* th is theta, the angle of rotation
  of the beam.  When the beam is upright,
 th=0;  when it is horizontal (cocked) th=Pi/2*)

nstrands = 8; (*count forward and back strands *)
n = nstrands / 2; (*number of strands/2 should be even*)
l1 =.; s =.; t =.; l2 =.
af =.; ab =.; bf =.; bb =.

i = 0; k = 0;
Do[
i = (-2 * k + n + 1);
 af[k] = {-l1, t + s, i * s}; (* left anchor point*)
 ab[k] = {-l1, -t - s, i * s};
 bf[k] = {l1, t + s, i * s}; (* right anchor point*)
 bb[k] = {l1, -t - s, i * s},
{k, 1, n }];


(* forward strands point of contact with beam,
 nearly upright beam:*)
i = 0; k = 0;
Do[
i = -2 * k + ( n + 1);
pfbig[k, th_] =
  {0, l2 Sin[th] - i s Cos[th], l2 Cos[th] + i s Sin[th] };
Print[pfbig[k, th]],
 {k, 1, n}
];

{0, -3 s Cos[th] + l2 Sin[th], l2 Cos[th] + 3 s Sin[th]}

{0, -s Cos[th] + l2 Sin[th], l2 Cos[th] + s Sin[th]}

{0, s Cos[th] + l2 Sin[th], l2 Cos[th] - s Sin[th]}

{0, 3 s Cos[th] + l2 Sin[th], l2 Cos[th] - 3 s Sin[th]}
```

```
(* back strands point of contact with beam,
 nearly upright beam*)
i = 0; k = 0;
Do[
i = -2 * k + (n + 1);
pbbig[k, th_] =
  {0, -l2 Sin[th] - i s Cos[th], -l2 Cos[th] + i s Sin[th] };
Print[pbbig[k, th]],
 {k, 1, n}
];
```

{0, -3 s Cos[th] - l2 Sin[th], -l2 Cos[th] + 3 s Sin[th]}

{0, -s Cos[th] - l2 Sin[th], -l2 Cos[th] + s Sin[th]}

{0, s Cos[th] - l2 Sin[th], -l2 Cos[th] - s Sin[th]}

{0, 3 s Cos[th] - l2 Sin[th], -l2 Cos[th] - 3 s Sin[th]}

```
(*check these for vertical beam *)
Do[Print[pfbig[k, Pi / 2]]
, {k, 1, n}]
Do[Print[pbbig[k, Pi / 2]]
, {k, 1, n}]
```

{0, l2, 3 s}

{0, l2, s}

{0, l2, -s}

{0, l2, -3 s}

{0, -l2, 3 s}

{0, -l2, s}

{0, -l2, -s}

{0, -l2, -3 s}

```
(* point of contact of strands with the beam,
 approaching horizontal, sm=small th*)
i = 0; k = 0;
Do[
i = 2 * k - 2;
pfsm[k, th_] =
  {0, t + s - s Sin[th] + i s Cos[th],
   l2 / Cos[th] - (s + t - s Sin[th] + i s Cos[th]) * Tan[th]};
Print[pfsm[k, th]],
 {k, 1, n}
];
(* back strands*)
i = 0; k = 0;
Do[
i = -2 * k + 2 n;
  pbsm[k, th_] = {0, -(t + s - s Sin[th] + i s Cos[th]),
   -(l2 / Cos[th] - (s + t - s Sin[th] + i s Cos[th]) * Tan[th])};
  Print[pbsm[k, th]],
 {k, 1, n}
];
```

```
{0, s + t - s Sin[th], l2 Sec[th] - (s + t - s Sin[th]) Tan[th]}

{0, s + t + 2 s Cos[th] - s Sin[th],
 l2 Sec[th] - (s + t + 2 s Cos[th] - s Sin[th]) Tan[th]}

{0, s + t + 4 s Cos[th] - s Sin[th],
 l2 Sec[th] - (s + t + 4 s Cos[th] - s Sin[th]) Tan[th]}

{0, s + t + 6 s Cos[th] - s Sin[th],
 l2 Sec[th] - (s + t + 6 s Cos[th] - s Sin[th]) Tan[th]}

{0, -s - t - 6 s Cos[th] + s Sin[th],
 -l2 Sec[th] + (s + t + 6 s Cos[th] - s Sin[th]) Tan[th]}

{0, -s - t - 4 s Cos[th] + s Sin[th],
 -l2 Sec[th] + (s + t + 4 s Cos[th] - s Sin[th]) Tan[th]}

{0, -s - t - 2 s Cos[th] + s Sin[th],
 -l2 Sec[th] + (s + t + 2 s Cos[th] - s Sin[th]) Tan[th]}

{0, -s - t + s Sin[th], -l2 Sec[th] + (s + t - s Sin[th]) Tan[th]}
```

```
(* check for horizontal beam*)
Do[Print[pfsm[k, 0]], {k, 1, n}]
Do[Print[pbsm[k, 0]], {k, 1, n}]
```

{0, s + t, l2}

{0, 3 s + t, l2}

{0, 5 s + t, l2}

{0, 7 s + t, l2}

{0, -7 s - t, -l2}

{0, -5 s - t, -l2}

{0, -3 s - t, -l2}

{0, -s - t, -l2}

```
(* input the parameters here,
 then calculate angle at which the attachment points
  interfere with the movement of the strands*)
param = {l2 -> 2, l1 -> 10., s -> 1., t -> 1.}
(*param={l2-> 11.4,s-> 3,t-> 7}*)
(*limit on movement of strand*)
solz =.
 phz =.
sol = Solve[Cos[phz] ==
    (t + s + (n - 1) * s * Sqrt[1 - Cos[phz] * Cos[phz]]) / (l2 + s),
  phz];

k = 4; (* search solutions for correct one*)

solz = phz /. sol[[k]](* the limiting angle for phi*)
the = Pi / 2 - phz /. sol[[k]] /. param
(* the limiting angle for th*)

(* the
  means th when top strand hits support, extremal*)

thdeg = (180 / Pi) * the
```

{l2 → 2, l1 → 10., s → 1., t → 1.}

Solve::ifun : Inverse functions are being used by Solve,
so some solutions may not be found.

$$\mathrm{ArcCos}\left[\frac{l2\,s + s^2 + l2\,t + s\,t + 3\,s\sqrt{l2^2 + 2\,l2\,s + 9\,s^2 - 2\,s\,t - t^2}}{l2^2 + 2\,l2\,s + 10\,s^2}\right]$$

1.27628

73.1255

```
(*   check that the two solutions
  are continuous at the (theta extreme)  *)
Do[
Print[pfsm[i, the] /. param];
Print[pfbig[i, the] /. param]; Print[],
 {i, 1, n}]
```

```
{0, 1.04306, 3.45138}

{0, 1.04306, 3.45138}


{0, 1.62361, 1.5375}

{0, 1.62361, 1.5375}


{0, 2.20416, -0.37639}

{0, 2.20416, -0.37639}


{0, 2.78471, -2.29028}

{0, 2.78471, -2.29028}
```

```
(*check back strands; pairs should be equal*)
Do[
Print[pbsm[i, the] /. param];
Print[pbbig[i, the] /. param]; Print[],
 {i, 1, n}]
```

{0, -2.78471, 2.29028}

{0, -2.78471, 2.29028}

{0, -2.20416, 0.37639}

{0, -2.20416, 0.37639}

{0, -1.62361, -1.5375}

{0, -1.62361, -1.5375}

{0, -1.04306, -3.45138}

{0, -1.04306, -3.45138}

```
(*Combine these for the
  general case where beam is at angle th*)
(* boole=True means restricted movement*)
boole =.
pfn[k_, th_, boole_] := If[boole, pfsm[k, th], pfbig[k, th]];
pbn[k_, th_, boole_] := If[boole, pbsm[k, th], pbbig[k, th]];
```

```
(* The unit vectors
  in direction of tension; L=left; R=right*)
Do[
TfL[k_, th_, b_] := Simplify[(pfn[k, th, b] - af[k]) /
    Sqrt[(pfn[k, th, b] - af[k]) . (pfn[k, th, b] - af[k])]];
TfR[k_, th_, b_] := Simplify[(pfn[k, th, b] - bf[k]) /
    Sqrt[(pfn[k, th, b] - bf[k]) . (pfn[k, th, b] - bf[k])]];
TbL[k_, th_, b_] := Simplify[(pbn[k, th, b] - ab[k]) /
    Sqrt[(pbn[k, th, b] - ab[k]) . (pbn[k, th, b] - ab[k])]];
TbR[k_, th_, b_] := Simplify[(pbn[k, th, b] - bb[k]) /
    Sqrt[(pbn[k, th, b] - bb[k]) . (pbn[k, th, b] - bb[k])]],
 {k, 1, n}
];
```

```
(* forces coming from each strand*)
Do[
forcefn[k_, x_, b_] := TfL[k, x, b] + TfR[k, x, b];
forcebn[k_, x_, b_] := TbL[k, x, b] + TbR[k, x, b],
  {k, 1, n}
];
rfn[k_, th_, b_] := pfn[k, th, b];
rbn[k_, th_, b_] := pbn[k, th, b];
(*torque/tension add in pairs;
 note the tension is a function of angle; *)

torqueovten[k_, th_, b_] :=
 CrossProduct[rfn[k, th, b], forcefn[k, th, b]] +
  CrossProduct[rbn[k, th, b], forcebn[k, th, b]];

torqtotovten[th_, b_] :=
 Sum[torqueovten[k, th, b], {k, 1, n}];
torqueovtenstot[th_] := If[th < the,
   torqtotovten[th, True], torqtotovten[th, False]][[1]];
```


```
(*torque/tension add in pairs;
 note the tension is a function of angle; *)

torqueovten[k_, th_, b_] :=
 CrossProduct[rfn[k, th, b], forcefn[k, th, b]] +
  CrossProduct[rbn[k, th, b], forcebn[k, th, b]];

torqtotovten[th_, b_] :=
 Sum[torqueovten[k, th, b], {k, 1, n}];
torqueovtenstot[th_] := If[th < the,
   torqtotovten[th, True], torqtotovten[th, False]][[1]];
```

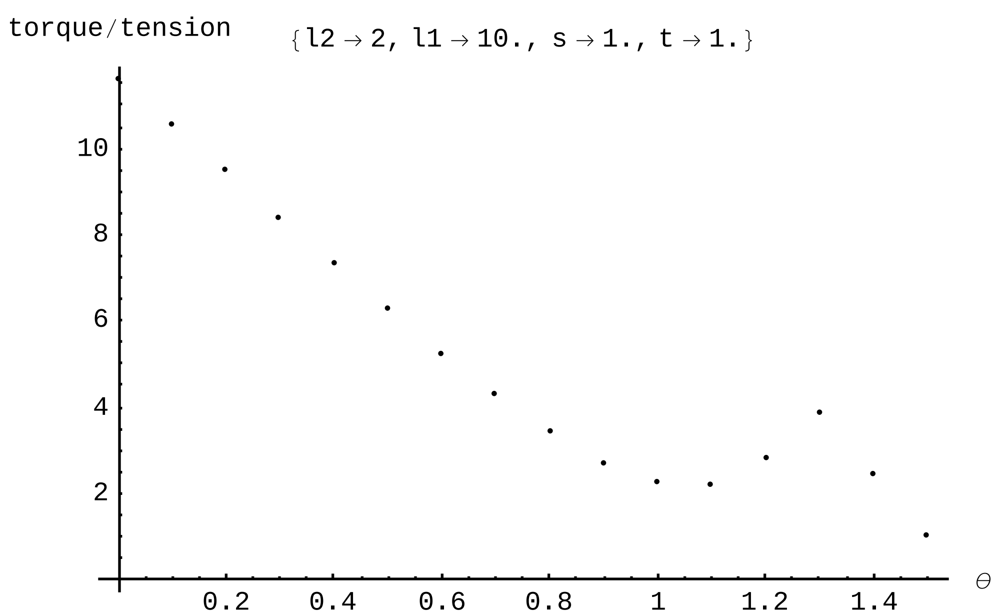
```
(*check these:*)
j = 4;
torqueovten[j, Pi / 2, False](* should be zero*)
forcefn[j, Pi / 2, False](* should be only along y axis*)
Simplify[forcefn[j, Pi / 2, False] + forcebn[1, Pi / 2, False]]
(* should be zero*)
```

{0, 0, 0}

$$\left\{0,\ \frac{2\,(l2-s-t)}{\sqrt{l1^2+(-l2+s+t)^2}},\ 0\right\}$$

{0, 0, 0}

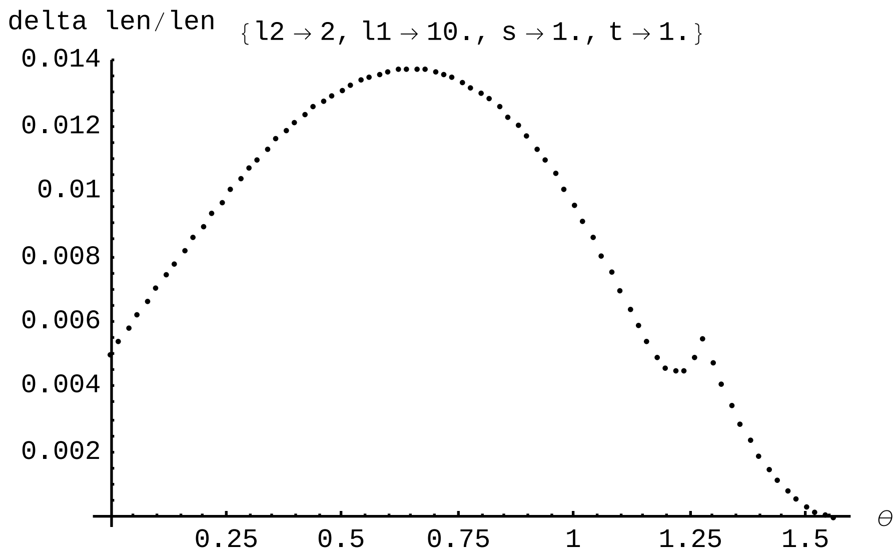
```
ListPlot[Table[{th, torqueovtenstot[th] /. param},
  {th, 0, Pi / 2, .1}], PlotLabel -> param,
 AxesLabel -> {"θ", "torque/tension"}]
```



- Graphics -

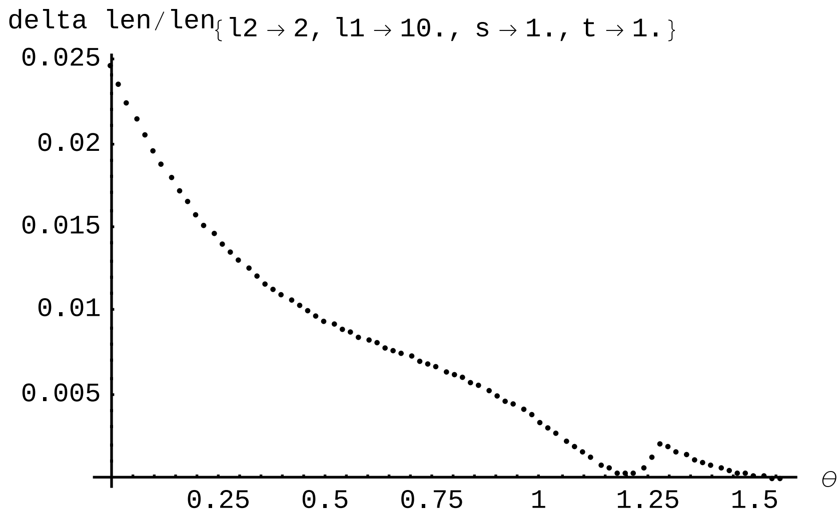
```
(*
 There are two possibilities:   the tension may
    be unequal in the different strands if the
    attachment points at the ends of the strands
    have high friction.  On the other hand,
 if there is no friction, the tension is the same in
  all strands:  the length changes experienced by the
    strands is distributed over all of the strands.*)

(* length of the kth forward strand: *)
lenstrand[k_, th_, b_] :=
 Sqrt[(pfn[k, th, b] - af[k]) . (pfn[k, th, b] - af[k])];
lenstrandtot[k_, th_] := If[th < the,
  lenstrand[k, th, True], lenstrand[k, th, False]];

luprt[k_] := lenstrandtot[k, Pi / 2]; (* starting length*)

frlenchng[k_, th_] :=
 (lenstrandtot[k, th] / luprt[k] - 1.) /. param
```

```
(* assume high friction at attachment points A and B
  to see how the length changes for each strand*)
ListPlot[
 Table[{th, frlenchng[1, th] /. param}, {th, 0, Pi/2, .02}],
 PlotLabel -> param, AxesLabel -> {"θ", "delta len/len"}]
```



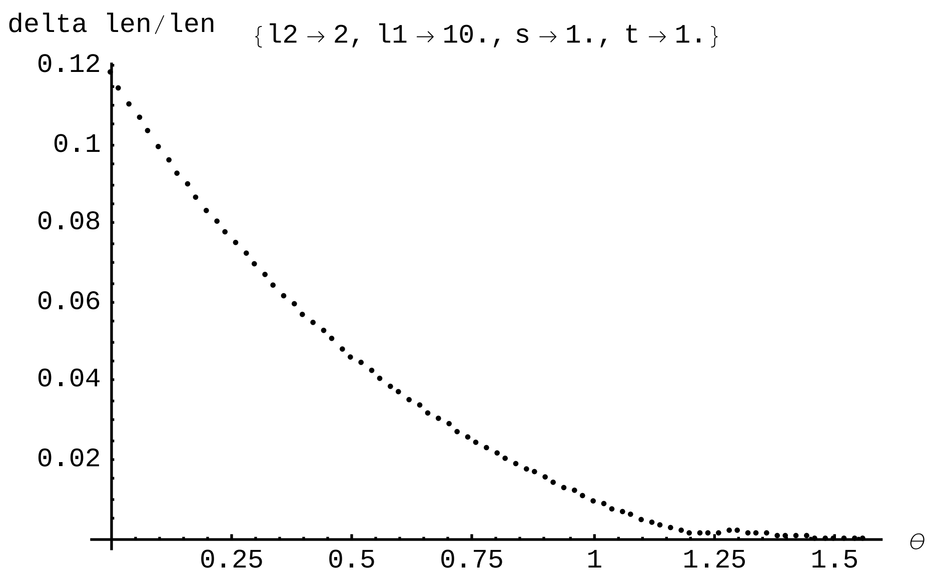
- Graphics -

```
ListPlot[
 Table[{th, frlenchng[2, th] /. param}, {th, 0, Pi/2, .02}],
 PlotLabel -> param, AxesLabel -> {"θ", "delta len/len"}]
```



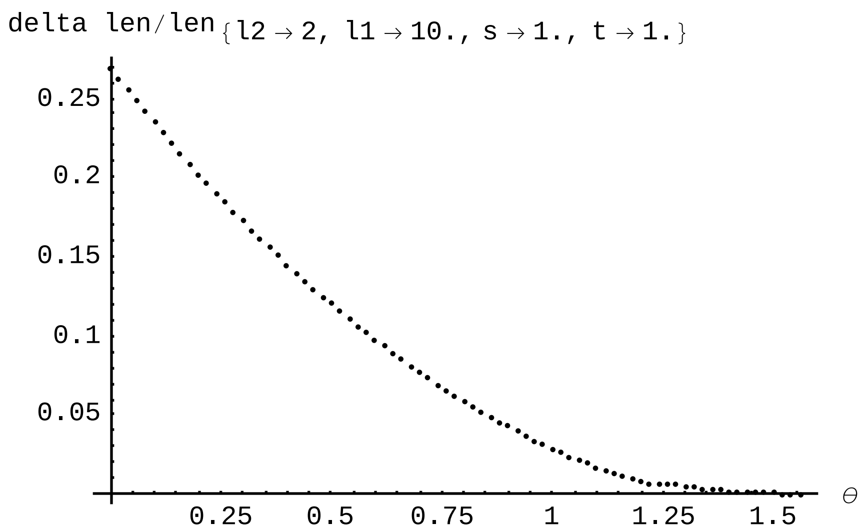
- Graphics -

```
ListPlot[
 Table[{th, frlenchng[3, th] /. param}, {th, 0, Pi/2, .02}],
 PlotLabel -> param, AxesLabel -> {"θ", "delta len/len"}]
```



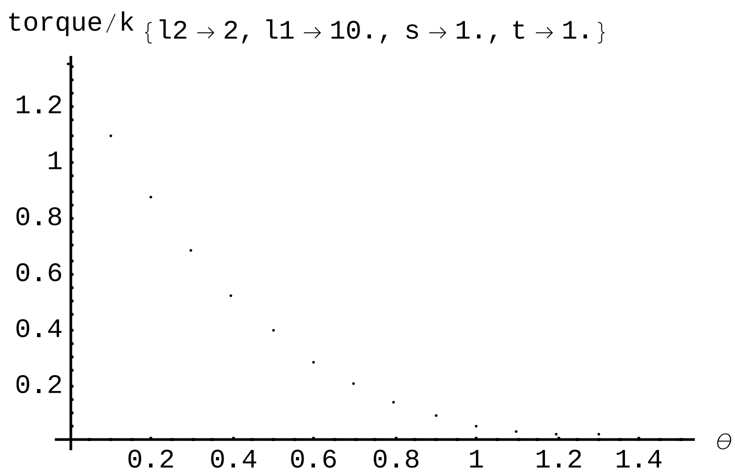
- Graphics -

```
ListPlot[
 Table[{th, frlenchng[4, th] /. param}, {th, 0, Pi/2, .02}],
 PlotLabel -> param, AxesLabel -> {"θ", "delta len/len"}]
```




```
- Graphics -
(*now assume that the tension
  in the strands are equal, independent of k*)
torq[th_, b_] :=
 Sum[torqueovten[k, th, b] * frlenchng[k, th], {k, 1, n}];
torquetot[th_] :=
 If[th < the, torq[th, True], torq[th, False]][[1]];
```

```
ListPlot[
 Table[{th, torquetot[th] /. param}, {th, 0, Pi / 2, .1}],
 PlotLabel -> param, AxesLabel -> {"θ", "torque/k"}]
```



- Graphics -

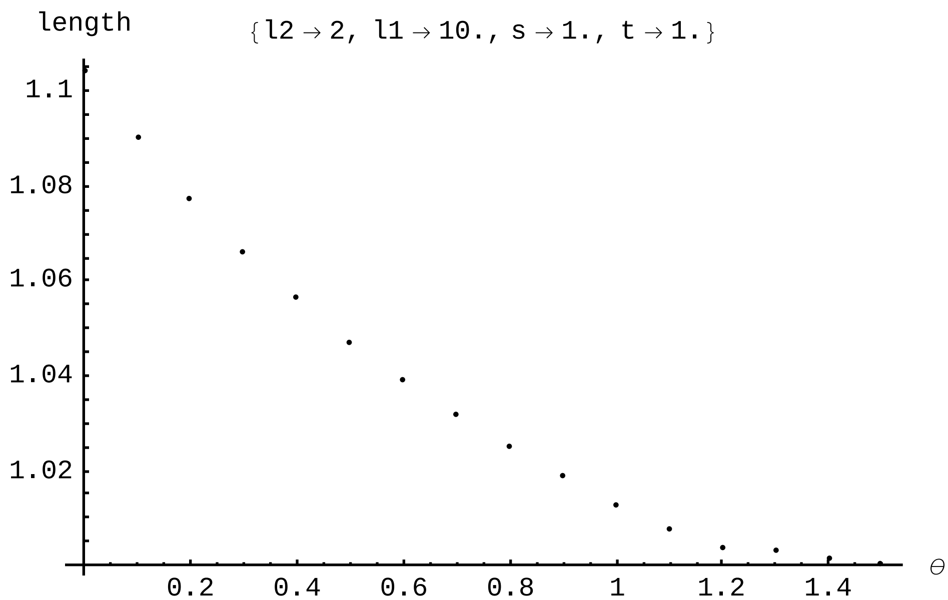
```
lensum[th_] := Sum[lenstrandtot[k, th], {k, 1, n}];
lensum[0](* length when beam is horizontal*)
```

$$\sqrt{l1^2 + (l2 - 3\,s)^2} + \sqrt{l1^2 + (l2 - s)^2 + 4\,s^2} + \sqrt{l1^2 + 16\,s^2 + (l2 + s)^2} + \sqrt{l1^2 + 36\,s^2 + (l2 + 3\,s)^2}$$

```
lensum[Pi / 2](* length when beam is vertical*)
```

$$4\sqrt{l1^2 + (l2 - s - t)^2}$$

```
(*length change for all
  the strands as a function of th.  Assumes
  all the strands are under equal tension*)
ListPlot[Table[{th, (lensum[th] / lensum[Pi / 2]) /. param},
  {th, 0, Pi / 2, .1}], PlotLabel -> param,
 AxesLabel -> {"θ", "length"}]
```



- Graphics -