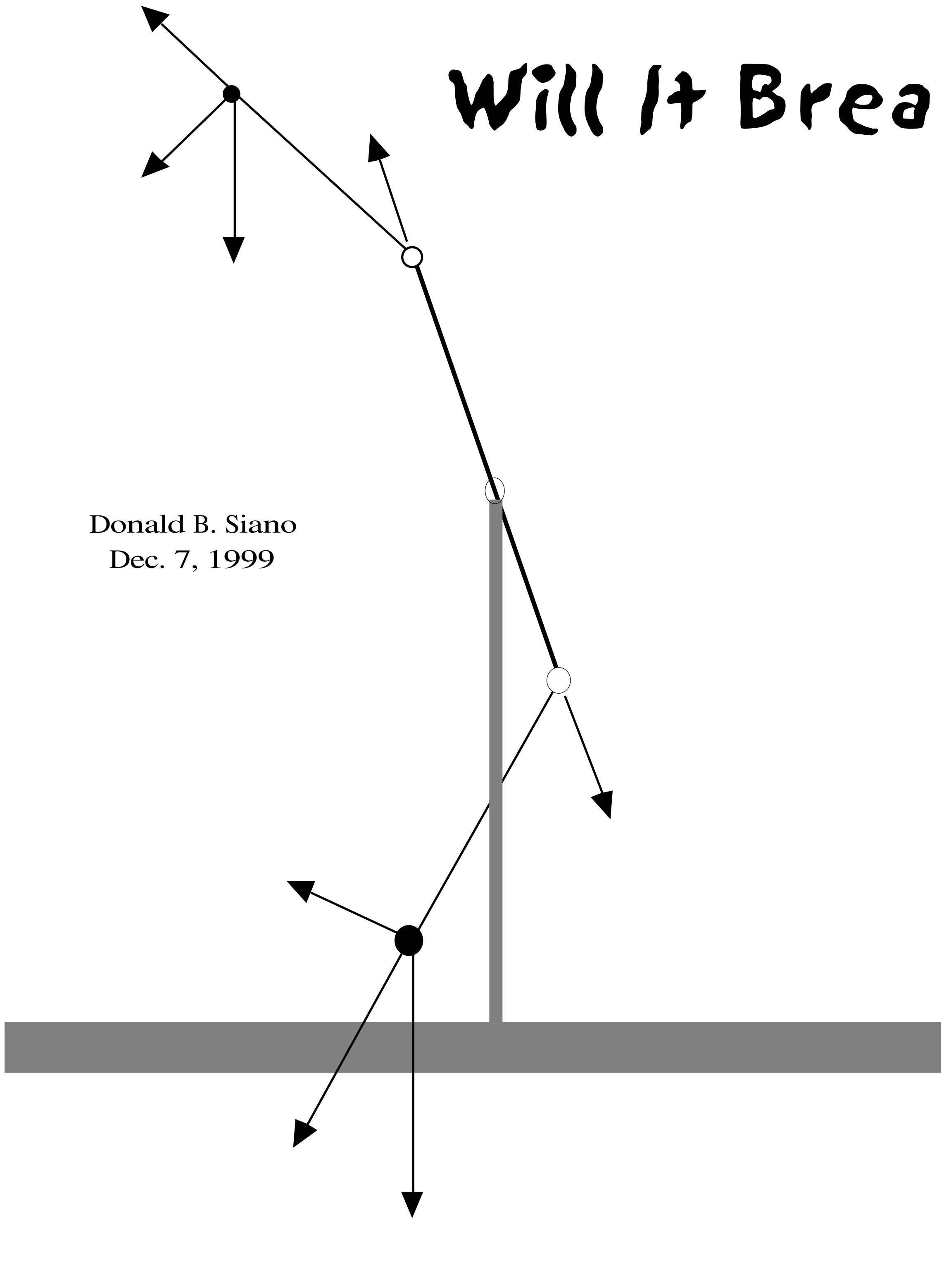

# Will It Break?

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## Introduction

One of the more critical aspects of designing a trebuchet is getting the components to survive the shock of launching the projectile. Breakage of any part of the mechanism means wasted effort in the construction and can be of grave danger to innocent bystanders. It would be desirable, therefore, to be able to do some engineering calculations that would allow one to estimate the strength of the parts that are most vulnerable to catastrophic disassembly.

There are two aspects to this problem. The first is to determine the forces and stresses applied to the various parts of the trebuchet during the throw. The second is to determine the strength of the parts as constructed, and whether the forces can be safely withstood. This latter task includes the determination of the relevant material properties used in the construction.

There is a certain amount of interaction between these two problems. For example, in a planned construction, one might assume a certain distribution of weight in the beam, and therefore be able to calculate the forces on the various parts of the trebuchet at all times during the throw by means of a simulation program. One would then use some engineering calculations to determine if each part is strong enough. If not, then the dimensions of the weak parts might be increased, which would yield changes in the characteristics of the throw and changes in the forces involved. Then the process might be repeated until all of the parts are strong enough, and the trebuchet has a desired high efficiency.

Therefore, one needs some general principles for design of the various parts, methods of calculating the forces, and perhaps, some rules of thumb or guidelines for design.

A diagram of the trebuchet in its starting position, showing the nomenclature used for the various lengths, angles and masses involved is shown in Appendix I. This is the same as that used in the manuscript "Trebuchet Mechanics" which describes the simulation of the trebuchet and gives some guidance in its design. That manuscript is very helpful in understanding some of the material in this document, and should be consulted if you are unfamiliar with the workings of a trebuchet. It is available on the "Algorithmic Beauty of the Trebuchet" site at www.members.home/dimona.

Our treatment of the problem will generally be fairly elementary--it will show how to apply some simple formulas to the various parts. Knowledge beyond elementary algebra will not be required, though some calculus is useful in understanding a couple of the derivations. Finite element analysis would allow a much more complete understanding of some of the various problems, but is taken to be beyond the scope required for our purposes. The first part of the discussion will assume that the forces are known by some means, and will focus primarily on the formulas involving the "strength" side of our problem.

## A Word of Caution

Recognize at the outset that all of the calculations to be described have some important limitations. First of all, the properties of wood vary considerably from one sample to the other, and with its grade, size and condition. Second, the forces calculated are derived from a simulation that is an idealization--all of the parts are assumed to be perfectly stiff, and friction is assumed to be negligible. Third, the stress formulas used themselves are based on idealizations and approximations--they may assume, for example, that the loads are static, and (usually) applied at points, rather than being distributed over finite areas; and constraints are idealized. Taken together, these three approximations mean that the following considerations should be used with due caution. They are much better than guessing, but they are not infallible. Allow yourself some margin for error, try to calculate how large errors may be for the situation under consideration, and use an appropriate safety factor in the end.

## The Axle

To start with, we will consider perhaps the most frequent source of problems: the bending or breaking of the axle about which the beam rotates. Nothing is more discouraging than to have a trebuchet completely constructed (finally!) only to find that after one or two throws the axle is deformed to the point that the fun stops.

We first model the axle and the forces acting on it as shown in Fig. 1. The axle is assumed to be a beam with length L, freely supported on the ends, with a load P in the center of the beam. The load comes primarily from the weight of the counterweight plus the centrifugal force generated by its motion during the throw. We will show how to estimate P for a given configuration of the trebuchet a little later.

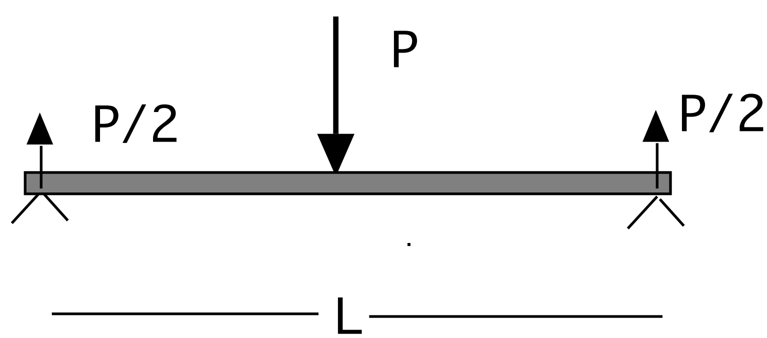



Fig. 1. A simply supported beam under a load P.

The axle will be bent and be permanently deformed when it is stressed past a certain point, which depends upon its shape and what it is made of. In modern trebuchets the material is most likely to be mild steel, while the medieval constructions used wood. Most axles are round, having a radius r.

To start with, as a crude approximation, we will replace the cylindrical axle or radius r, with one having a cross-section that is a square with sides 2r. Intuitively, this will be somewhat stronger (having more load-bearing material) than the actual one, but results in a slightly more simple calculation.

Again, for the sake of simplifying the problem to something we can readily solve, we will replace the realistic, complicated problem of strength under a dynamic load, with a problem in statics. That is, we neglect the fact that the force P is only momentarily applied, and is constantly changing in magnitude and direction during the throw. This will result, clearly, in an overestimate in the required strength of the axle, in the opposite direction caused by replacing the circular with the square cross-section. There will be some cancellation of error here.

So now we can use some standard results from elementary beam theory, which can be found in any text-book on statics and strength of materials. For the sake of completeness in the exposition we'll briefly go through the reasoning ourselves. If you are not interested in this derivation, jump to the next section, on the flexural formula, to learn how to apply the results by a "plug in the numbers approach"

First of all, note that there will be a reaction force at the two points of support of the axle, and by symmetry they each have a magnitude P/2.

We now imagine a cut perpendicular to the axle to the left of the point where the load is applied, at a distance x from the left hand point of support. The axle is at equilibrium, so the sum of the forces on it must be zero, and the sum of the torques about any point must be zero. Since we know there is a reaction force at the left end, there must be other forces and torques (moment of forces) acting on it to keep it in equilibrium. A little reflection will reveal that just a force, or just a torque, will not enable the section to be in equilibrium. We assume that the force and moments are acting at the section of the cut--they must come from the other part of the beam and be transmitted at the cut section. The force, called the shear force, V, acting on the axle, which may be a function of position of the cut, will be as shown in Fig. 2.

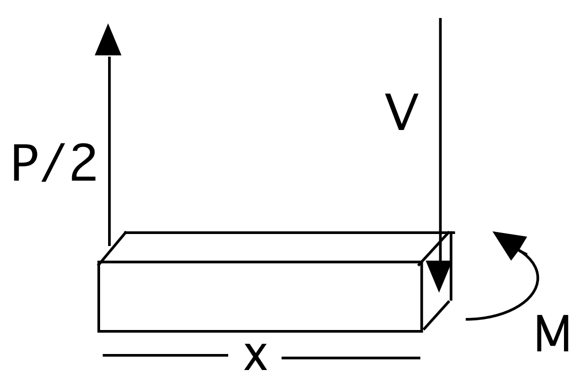



Fig. 2. The cut beam showing the shear force V and the torque M exerted on it by the right part.

The forces sum to zero because the axle is not accelerating vertically. That is,

$$P/2 - V = 0,$$

where V is the magnitude of the shear force.

We now consider the moments of the forces about the left hand support. A moment of force about an axis is the product of the magnitude of the force and the distance from the axis, multiplied by the sin of the angle between them. The moment of the shear force is clearly V x sin(90) = V x, and since it tends to rotate the beam clockwise, using the usual sign convention, it is negative. When the axle (beam) is under load, it should be clear that the bottom of the axle is under tension (the internal forces are acting to pull it apart) and the top is under compression. In the middle of the beam there is a line (the neutral line) where the material is neither compressed nor pulled. This gives a picture of the forces acting on the interior of the axle as shown in Fig. 3.

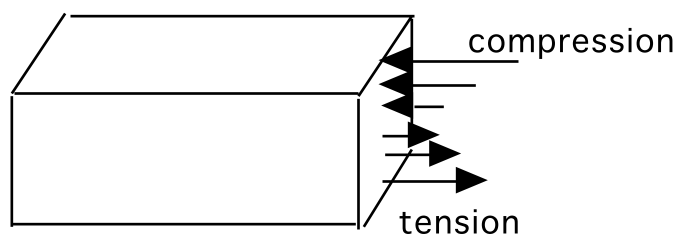



Fig. 3. The forces of compression and tension acting on the cut beam to give the bending moment.

These forces when added together will generate a torque or moment of force (a force times a distance) about the neutral line, called the bending moment, which will also tend to rotate the axle. Since the axle doesn't rotate around the end, these moments will sum to zero. That is,

$$M - V x = 0.$$

Solving these two equations simultaneously for the unknowns, V and M, gives immediately

$$V = P/2$$

and

$$M = P/2 \ x.$$

An analysis to the right of the center can be made in a similar way, and results in graphs of the shear force and shear moment as a function of x, the distance from the left-hand support, and is shown in Fig. 4.

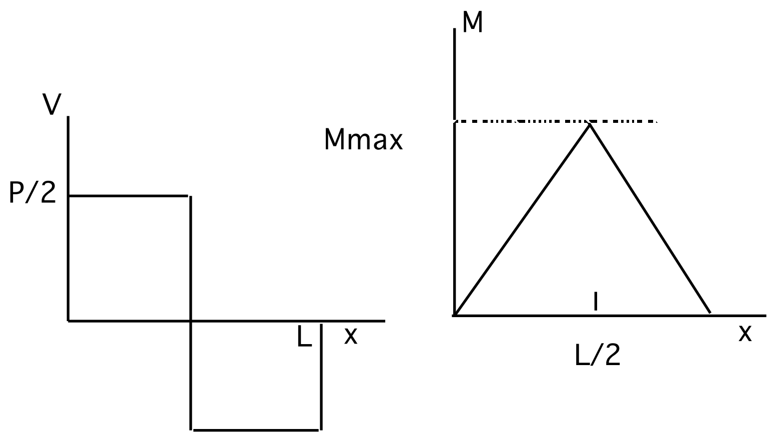



Fig. 4. The shear force and bending moments in the axle as a function of x.

Clearly the moment M has a maximum of L P/4 at the point of application of the load, where x = L/2, This (bending) moment is larger for larger loads and longer axles, and so partly determines whether it deforms or not.

## Flexure formula

At this point I'll invoke (without deriving it, though it is fairly simple to do) the flexure formula which is the central equation that is used to determine the outcome of the experiment. A thorough understanding of it is an absolute requirement for making your trebuchet safe.

It says that the stress, $\sigma$, in the beam (axle) measured at a point a vertical distance y from the middle of the beam is given by

$$\sigma = M\, y/I,$$

where I is the moment of inertia of the section of the beam. The stress at each point in the cross-section is the quantity that tells us how strong the forces tending to deform the beam are. The stress has the units of force per unit area, and therefore is similar, in a way, to a pressure, which has the same units. According to the formula, it is greatest at the outer edge of the beam where y is a maximum. Therefore failure of the axle will first occur there. On the top of the axle it will fail by compression, and on the bottom by tension. Every material has a maximum allowed stress, which we can look up in a table somewhere and compare it with our calculated stress.

The maximum moment Mmax (M at ymax) is known from the discussion we just went through--Mmax = L P/4, where L is the length of the beam and P is the load, assumed to be concentrated at the center, in the problem under consideration. M, y, and I, of course, vary for different situations.

The moment of inertia of the section, I, is not difficult to calculate. In general, it is calculated by multiplying an infinitesimal area of the section by the square of its distance from the neutral axis, and integrating over all of the area. Referring to Fig. 5, this is calculated for a square section as shown in Fig. 5.

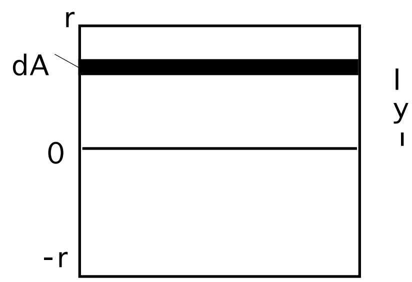



Fig 5. The moment of inertia of a beam with a square cross-section about the neutral axis.

Since the infinitesimal area, dA, is 2r dy, we get for a beam with a square cross-section

$$I = \int_{-r}^{r} y^2 \, dA = \int_{-r}^{r} y^2 \, 2r \, dy = \frac{4\, r^4}{3}$$

We see from this that the moment of the section has a dimension of length to the fourth power. The moment of inertia of sections with different shapes can be readily calculated, or, in the absence of knowledge of integral calculus, just looked up in engineering tables. For a circular cross-section of radius r, the moment of inertia of the section is $(\pi\, r^4)/4$.

Gathering these results together, we find that the maximum stress (at the outer perimeter of the square axle) is therefore

$$\sigma_{max} = M_{max}\, r/((4\, r^4)/3)$$

$$= (L\, P/4)(3/(4r^3),$$

or, finally,

$$\sigma_{max} = (3/16)\, P\, L\, /r^3, \quad \text{(square cross-section).}$$

For a circular cross-section, it is easy to verify that the corresponding formula has a coefficient of $(1/\pi)$ instead of 3/16. That is,

$$\sigma_{max} = (1/\pi)\, P\, L\, /r^3, \quad \text{(circular cross-section).}$$

The cylinder is $3\pi/16$ = 59% as strong as a square cross-sectional axle.

For a pipe with outer diameter $r_1$ and inner radius $r_2$, we get

$$\sigma_{max} = (1/\pi)\, P\, L\, r_1/(r_1^4 - r_2^4), \quad \text{(hollow cylinder).}$$

To use the flexural equation is very simple: plug in your values for P, L and r to get the maximum stress, and compare it with the maximum allowed stress for the material used for the axle. If the calculated stress is larger than that which the material can support, then it will bend. For mild steel, for example, the tensile strength allowed might be taken as 36,000 psi.

This equation is seen to be at least directionally correct. When the length of the axle is longer or the load P is higher, $\sigma_{max}$ is larger by the same factor, and we need stronger material. When the radius r is twice as large, the axle is eight times stronger.

Example 1. Suppose the load P on a cylindrical axle is 1000 pounds, it has a radius of 1 inch and the distance between supports is 2 ft. It is made out of mild steel.
Will it break?

Answer:
Use the appropriate flexural formula. Then

$$\sigma_{max} = (1/\pi) \times 1000 \times 24/1^3 \text{ pounds/in}^2$$

$$= 7600 \text{ pounds/in}^2$$

This is less than the allowed yield strength of 36,000 psi by a comfortable margin, so it should not bend. If the radius of the beam is 0.5 in, the bending stress will be eight times as great, and therefore unsafe for this load.

## Units

Example 1 employed the American system of units, where force is in pounds, stress is measured in pounds per square inch, and moments of inertia are in $in^4$. The rest of the world may more commonly use the unit of force as Newtons, and the unit of stress as meganewtons per square meter, $MN/m^2$. The conversion factor is 1 psi = 0.00685 $MN/m^2$. The strength of mild steel is therefore about 250 $MN/m^2$. A pound of force is 4.45 N, and an inch is 2.54 cm. Sometimes the units of stress given are in Pascals: one Pascal (Pa) is a Newton per square meter.

Using these units in Example 1, would then give

$$\sigma_{max} = (1/\pi) \times 4450 \times 24 \times 2.54/2.54^3 \text{ N/cm}^2$$

$$= 53 \text{ MN/m}^2,$$

which is comfortably below the limit of 250 $MN/m^2$, so we do not expect it to bend or break.

## Strength of Materials

Knowledge of the strength of the materials that might be used in trebuchets is, of course, essential for arriving at an efficient design. There are many trade-offs involved in materials that might be used; we are concerned among other things with its strength per unit weight for a given sort of load, its cost, "authenticity" and availability.

There are, of course, a number of different measures of the strength of materials, some of which are more important for our purposes than another. And there are two very different sorts of materials: metal and wood. While there are entire books written on these subjects, we can make do with some simple ideas at first.

### Metals

Metals are somewhat ductile--that is, they will yield (stretch) before they break. In a trebuchet axle, for example, we would be interested in knowing the load under which it will begin to permanently deform. The appropriate material property is its "yield strength". We are not so interested in the point at which the axle actually breaks, correlated with the "ultimate strength", sometimes called the "tensile strength". These are very different numbers for metals. When referring to published tables, one should be sure to make this simple distinction. A further point of confusion is that every structural material has different grades, each with a different tabulated strength. If you don't know the grade of the material, you won't have an accurate measure of its strength. But one can usually still make reasonable assumptions.

For steel, there are many grades. But an awful lot of stuff is made out of "mild steel" with a yield strength of about 36,000 psi or 250 MPa. A hardened steel would have a yield strength of 50,000 to 75,000 psi. Heat treated stainless steel might go as high as 132,000 psi. We'll use mild steel in the examples.

Aluminum alloys range from about 40,000 to 70,000 psi. When considered in terms of yield, they are actually about as strong as mild steel.

For bending or stretching of a beam or other structural member due to loads below the yield point, where the material elastically deforms, we need the Young's modulus (modulus of elasticity). This has the same units as yield strength, but a different magnitude. For practically any class of carbon steel it is about 30,000,000 psi.

For metals, the yield strength is generally the same for loads under tension as for compression.

### Wood

The material properties for wood are considerably more complex. Wood has a grain, so its properties are different in different directions. There are many more kinds and grades of wood available than for metals. It is impossible in a document such as this one to be particularly thorough in discussing each important aspect of the strength of wood. We can only hit a few of the high points, and try to point out some of the more relevant considerations. Rules of thumb can help.

Perhaps the most important mechanical property of wood for our purposes is its modulus of rupture. This is the stress that it can withstand before it breaks, and is a sort of analog of the yield strength of metals. For various woods, it ranges over something like 5000 to 20,000 psi. It is generally measured in a test involving static bending.

For elastic bending of parts such as beams, like metals, one uses a modulus of elasticity, which ranges from about 1 to 2 million psi.

Tabulated values of the rupture strength of wood refer to small "clear" samples, which have no knots, voids, cracks or other imperfections one would find in actual pieces found in a lumber yard. One therefore needs to discount the tabulated strength of the wood by an amount that depends principally on the grade of the wood. For example, structural joists and planks grades range from "select structural" to grades 1, 2, 3 and 4, having rupture strength ratios of 0.65, 0.55, 0.45, and 0.26. Thus, for example, the lowest grade has a rupture strength of only 26% of the tabulated "clear" values.

The strength of wood depends upon the drying of the wood. Green wood has a rupture strength about 30% lower than properly dried wood.

Another consideration for trebuchets is that the forces are mostly applied for only a short time. There exist correlations for the dependence of strength upon the duration of the load. One learns (see the references given at the end of this manuscript) that a beam can sustain a load about 20% greater if it is applied for 0.1 s than if it is applied for an hour.

Generally speaking, wood is weaker under compressive loads than in tension. If the compressive load is in the direction of the grain, it is something like only half as strong as the rupture strength.

Example 2:
Get the rupture strength of white oak that is second grade, with a load that is applied for a very short time.

Solution:
The tabulated rupture strength of dried white oak is 17,700 psi. It is second grade, so we use a value of 0.45*17,700 = 8000 psi. Assume the load is applied for 0.1 s, so it would be 20% stronger, or 9600 psi.

## The counterweight Box

The counterweight is sometimes designed as a simple box that hangs from the end of the beam. It can experience several failure modes. It can break at the point it connects to the beam, or the bottom can fall out of it, for example. How it fails has to be investigated for each design under consideration. We will consider one configuration, as illustrated in Fig. 6.

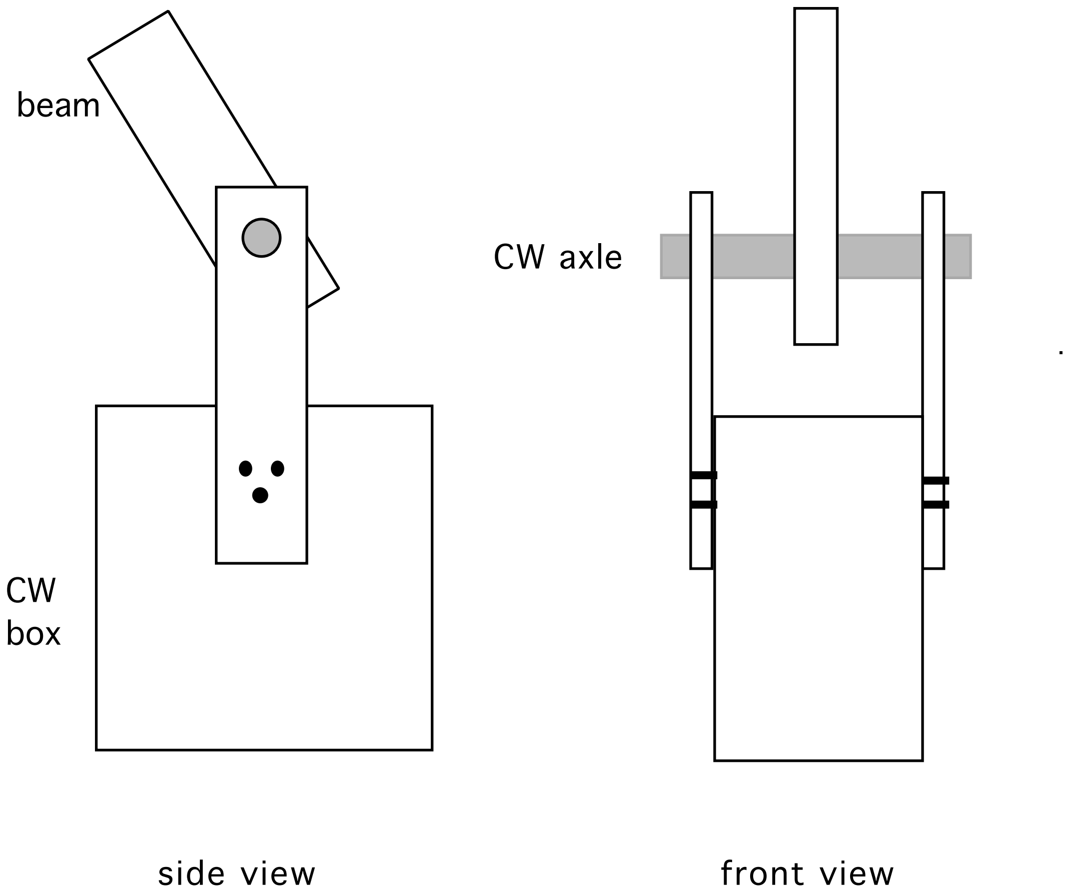



Fig. 6 The counterweight box and its hanger, shown connected to the beam through the CW axle.

CW axle

The box is suspended from an axle, which might bend. This is the same problem as already solved in example 1. There are two equal forces at the end of the CW axle, and a force in the opposite direction at the center. So we just use the same formula as before. Simple. If we make this CW axle out of the same bar of material used in the main axle, everything will certainly be well because the distance between the supports will be less, and the forces involved in bending it will be about the same.

CW Box Hanger

The box is sometimes designed to hang from the axle by two simple supporting members as shown in Fig. 6. If each is a rectangular parallelepiped with cross-sectional area $A_h$, and the box has a downward force on it of Fb, then the stress in the section of the hanger is clearly $\sigma = Fb/(2\ A_h)$. As an example, take the strength of the wood to be 1000 psi, and the force on the box to be caused by 1000 lbs of weight at 3 g's. Assume the hangers are two by fours.

These actually have dimensions of 1.5 x 3.5 in, so their area is $A_h = 5.25\ in^2$. Then the stress in each hanger would be 3000/(2 * 5.25) = 200 $p/in^2$. This is well below the rupture strength of wood, so the hangers are safe.

CW Box Bottom

How do we determine if the bottom will break? This is a little harder. If we suppose that the bottom of the box is made of planks with width w, thickness t, and length L, then we can make some headway by assuming the total load P in the box is distributed evenly along the planks. Suppose the plank is only supported at the two ends. Then we can apply the flexural formula to see if it breaks.

First, we need the moment of the force at a distance from the left end. If s is the force exerted on the plank per unit length (s has units of pounds of force per foot, or N/m), and is a constant, then the moment of force caused by the load is

$$M = s \int_0^x x\, dx = \frac{s\, x^2}{2}$$

The maximum moment of force is again at the midpoint of the plank where x = L/2, so

$$M = \frac{s}{8} L^2$$

The moment of inertia of the section of the plank is easy to get, using the approach of Example 1. It is

$$I = \int_{-t/2}^{t/2} w\, y^2\, dy = \frac{w\, t^3}{12}.$$

Putting these two results together in the flexural equation , we can get a formula that will tell us if the plank will break. Thus,

$$\sigma_{\mathbf{max}} = \frac{3\ \mathbf{s}\, \mathbf{L}^2}{4\ \mathbf{w}\, \mathbf{t}^2}$$

determines if the uniformly loaded plank will break: if the right hand side of the equation is greater than the tabulated tensile strength of the material, then we need to strengthen it. Sometimes the easiest method is to increase the thickness of the plank. As shown by the formula, doubling it will increase the strength by a factor of four.

Example 3: CW bottom

As an example of the application of this formula, suppose the box has a bottom that is 2 ft wide and uses planks 8'' wide and 0.75 in thick. The planks are then L = 2 ft long. The CW box carries a load of 1000 pounds and experiences a maximum of 3 g's in the direction perpendicular to the bottom. Then the maximum load is 3000 pounds on the three planks, or 1000 pounds per plank. The (uniform) load on each plank is then s = 1000/2 = 500 pounds per foot. Plugging in the numbers, converting all lengths to inches, gives

$$\sigma_{\mathbf{max}} = \frac{3\,(500/12)\,24^2}{4\;8\;0.75^2}$$

$$= 4000 \text{ pounds/in}^2$$

Comparing this to the values for the rupture strength of wood shows that this is quite marginal; better strengthen it.

## Estimating the static force on the axle

We'll start with our estimate of the forces on the axle by considering the forces when the beam is cocked in preparation for the throw. The configuration is then as shown in Fig. 7.

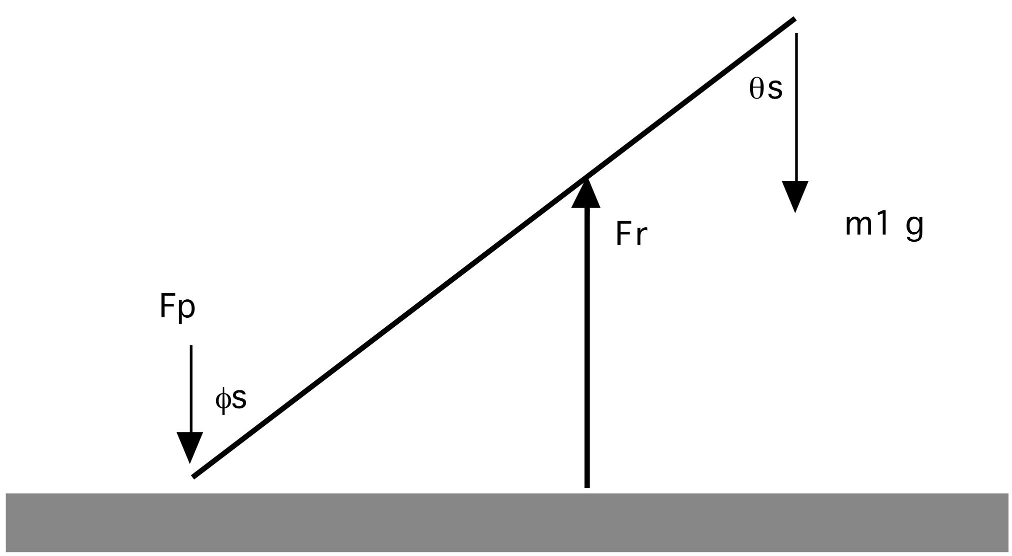



Fig. 7. The forces on the beam while pinned at the trigger end.

This is a problem in statics: There is a load on the short end of the beam caused by the counterweight with mass m1. It is at a distance l1 from the axle and is directed in the vertical direction. The forces on the beam are large, so we assume for simplicity that the beam is massless. The beam is prevented from rotating by being pinned at the projectile end at a distance l2 from the axle. The pin exerts an (unknown) force Fp on the beam. These forces result in reaction force, Fr, upward on the beam, exerted by the axle. We are interested in the magnitudes of these forces.

While there are several ways of pinning the left end of the beam, we will assume that the force applied has only a vertical component. This means, since g is also in the vertical direction, the reaction force will have no component in the horizontal direction and the problem is very simple. Solving the problem when Fp has a horizontal component is easy enough, and is left as an exercise for the reader.

Since the beam is not rotating or translating, and the forces are coplanar, we know that the sum of the forces is zero, and the sum of the torques about any point is zero. Summing the forces, we see that

$$m_1 g + F_p - F_r = 0$$

where the two unknown forces are defined so that they have components only in the "y" direction.

Summing the torques about the axle gives

$$l_1 m_1 g \sin \sin(\phi_s) - l_2 F_p \sin(\theta_s) = 0$$

Solving these two equations for the unknown forces, we readily obtain

$$F_r = m_1 g \left(1 + \frac{l_1}{l_2}\right)$$

and

$$F_p = m_1 g \frac{l_1}{l_2}$$

where we have used the fact, since the unknown forces are parallel, that $\sin(\theta_S) = \sin(\phi_S)$.

Thus, we see that the reaction force on the axle is a little greater than the weight of the counterweight (25% greater when l2/l1=4) and is independent of the starting angle.

## Bending and Breaking of the Beam

The independence of the pinning and reaction forces on the angle that the beam makes with respect to the horizontal implies that the same formulas apply when the beam is being cocked by pulling down on the sling and is horizontal, so $\sin(\theta_S) = \sin(\phi_S) = \pi/2$. The situation is like that shown in Fig. 8.

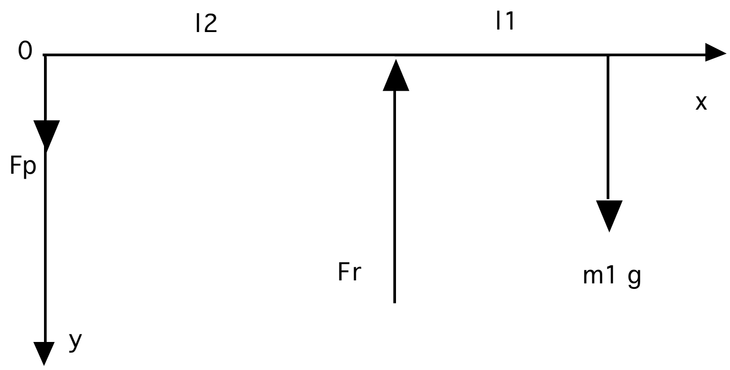



Fig. 8  The forces during the cocking when the beam is horizontal.

This is a configuration where the transverse forces on the beam are maximal, so the beam would exhibit the maximum amount of bending.  This is the condition that we will explore to see if the beam will break.   It is also in a configuration where it is relatively easy to solve for the shape that the beam takes.

We will assume that the support of the beam at the axle does not involve any hole in the beam through which the axle protrudes.  The reaction force at the axle is idealized to be exerted at a single point.

To the left of the axle, the bending moment is

$$M = F_p\, x = m_1\, g\, \frac{l_1}{l_2}\, x; \quad x < l_2$$

and to the right of the axle,

$$M = F_p\, x - F_r\,(x - l_2) = m_1\, g\, \frac{l_1}{l_2}\, x - m_1\, g(1+\frac{l_1}{l_2})\,(x - l_2); \quad x > l_2$$

For this condition, it is easy to see that the maximum moment occurs for x=l2--that is, at the axle.  Assume the beam has a width w and a thickness t, so it has a moment of inertia of w t^3/12, as previously shown.  The maximum stress is then found from the flexure formula, with y = t/2 to be

$$\sigma = \frac{6\, m_1\, g\, l_1}{w\, t^2}.$$

This formula can then be used to determine if the calculated stress exceeds the rupture strength of the wood used to make the beam.

Our object now is to get the amount of the bending of the beam as a function of the load, m1, the lengths l1, l2, the material the beam is made out of, and its cross-section. We will draw a graph of the shape of the beam under load, and try to judge whether the beam is dangerously weak.

We will start our analysis of the amount of bending of the beam by using a formula for the deflection, y, of the mid surface of the initially straight beam. The formula, which we do not derive here, is

$$Y\,I\frac{d^2y}{dx^2} = M$$

where I is the moment of inertia of the cross-section Y is Young's modulus, M is the bending moment of the cross-section of the beam. Young's modulus is the material property that describes how much it stretches for a given force per unit area. It is the proportionality factor between the stress (a force per unit area) and the fractional change in length, dl/l. The larger Y, the more resistant to stretching forces the material is. Young's modulus for wood is about $1.6 \times 10^6$ pound per square inch, or 11,000 $MN/m^2$.

To review the dimensions of the quantities we note: M has dimensions of Force x L, I has dimensions $L^4$, Y has dimensions of $F/L^2$, the derivative has dimensions of $L^{-1}$. The equation is therefore dimensionally correct. No surprise.

The solution to the differential equation is done for the two sections to the left and right of the axle.

The solution of the differential equation, subject to the conditions that y(l2)=0 and y'(l2) = 0 is, which one may readily verify by integration, on the left:

$$y_l = \frac{m_1 g}{Y\,I}\left[\frac{l_1}{6\,l_2}\,x^3 - \frac{l_1\;l_2}{2}\,x + \frac{l_1\;l_2^2}{3}\right]; \qquad 0 < x < l_2$$

and on the right is

$$y_r = \frac{m_1 g}{Y\,I}\left[-\frac{x^3}{6} + \frac{l_1 + l_2}{2}\,x^2 - \frac{2\,l_1 l_2 + l_2^2}{2}\,x + \frac{(3\,l_1 + l_2)l_2^2}{6}\right]; \qquad l_2 < x < l_1 + l_2$$

These two equations thus define the bending of the beam. Their utility can be illustrated by an example.

Example 4: A 2x4 trebuchet

Suppose we are planning to build a trebuchet with a beam that is made from a "2x4" 5 feet long. The 2x4 is oriented with the 4" side horizontal. We want to put a counterweight of 500 pounds on it. How much does it bend during cocking if the ratio of l1 to l2 is 4?

Answer
Put l1 =1 and l2=4 into the two equations to get

$$y_l = \frac{m_1 g}{Y I}\left[\frac{1}{24}x^3 - 2x + \frac{16}{3}\right]; \qquad 0 < x < 4$$

and

$$y_r = \frac{m_1 g}{Y I}\left[-\frac{x^3}{6} + \frac{5}{2}x^2 - 12x + \frac{56}{3}\right]; \qquad 4 < x < 5$$

where x is measured in feet. To convert the displacement to inches, we need to multiply the term in brackets in the two equations by the cube of 12, and be sure the constant term outside the brackets is also in units of inches (actually, $in^{-2}$).

Now "2x4"s actually have dimensions 1.5 x 3.5 in. To get the moment of inertia when the long side is horizontal we can use the formula given for the plank above, $I = w\,t^3/12 = (3.5 \times 1.5^3)/12 = 0.98\ in^4$. If $Y = 1.6 \times 10^6$ psi then we get a value of

$$m1\ g/(YI) = 500\ p/(1.6 \times 10^6\ psi \times 0.98\ in^4)$$
$$= 320 \times 10^{-6}\ in^{-2}.$$

Thus when x= 0 at the left end of the beam, the displacement is

$$y_1 = 320 \times 10^{-6}\ 16/3 \times 12^3 = 2.9\ in.$$

Similarly, on the right end, where x=l1+l2, we get $y_r = 0.18$ inches. Can a 5' long beam sustain bends of this magnitude? Probably so.

Suppose the 2x4's orientation was changed so that the long end is vertical. How much would y1 at the left end change? The moment of inertia would now be $1.5 \times 3.5^3/12 = 5.4\ in^4$. This is about 5.5 times larger, so the displacement is 5.5 times smaller than before. Makes a difference.

For this condition, it is easy to see that the maximum bending moment occurs for x=l2--that is, at the axle. Assume the beam has a width w and a thickness t, so it has a moment of inertia of w t^3/12, as previously shown. The maximum stress is then found from the flexure formula, with y = t/2 to be

$$\sigma = \frac{6\ m_1\ g\ l_1}{w\ t^2}$$

$$= 6*500*12/(2*1.5 \times 3.5^2)$$

$$= 1000\ p/in^2.$$

Suppose the 2x4 was made of eastern white pine, which has a rupture strength of 8,600 psi. If it is second grade, it could sustain 45% of this, or about 4000 psi. The beam is strong enough, confirming our opinion formed by looking at its shape while bent.

Returning to the shape of the beam, it is easy to put the formulas for yl and yr into a spread sheet and get the values as a function y, so that the bending may viewed to scale. It would look something like this:

Fig. 9. The shape of the beam during cocking when the beam is horizontal.

The beam bending equations are also derived in Appendix II using the Mathematica Programming language. It is a relatively simple matter to derive analogous equations for the case in which the moment of inertia increases linearly along the beam, but the expressions get a little unwieldy.

## Scaling of the forces

It is of interest to inquire how the forces tending to break apart the trebuchet scale with changes in length of the parts. That is, we ask, if all of the lengths are doubled in the trebuchet, what happens to the forces? Do they also double?

To answer this question, consider the accelerations involved. One argument uses the insight from the dimensional analysis of the problem as described previously in the "Mechanics of the Trebuchet." We learned that as the lengths doubled, the time for various events went as the square root of two. Since acceleration has dimensions of $L/T^2$, we would therefore expect the accelerations to be unchanged by a proportionate change in all of the lengths.

This is, in fact, what is actually found by using the simulation program, TrebMath 5.4, which calculates the accelerations of the CW and the projectile. Table II shows results for several cases as examples.

Table II. Results of several simulations used to investigate scaling properties of the accelerations.

Parameters used:

| | | | | |
|---|---|---|---|---|
| l1 (short) | 2 | 1 | 1 | 4 |
| l2 (long) | 8 | 4 | 4 | 2 |

| | | | | |
|---|---|---|---|---|
| l3 (sling) | 7 | 3.5 | 3.5 | 1.75 |
| l4 (CW arm) | 2 | 1 | 1 | 0.5 |
| l5 (axle ht) | 5.66 | 2.83 | 2.83 | 1.41 |
| m1 (CW) | 800 | 100 | 800 | 12.5 |
| m2 (proj) | 8 | 1 | 8 | 0.125 |
| mb (beam) | 54 | 6.75 | 54 | 0.85 |
| Output: | | | | |
| Range | 449 | 223 | 223 | 112 |
| Reff | 0.659 | 0.654 | 0.659 | 0.659 |
| a(proj) | 376 | 376 | 376 | 376 |
| a(CW) | 36 | 36 | 36 | 36 |

As expected, the range scales as the lengths while the range efficiency is a constant. The accelerations of the counterweight and projectiles are constant when all of the lengths are changed proportionately. Moreover, the accelerations for the projectile and CW are the same even when the masses are changed by a factor of eight. This is to be expected also--the acceleration of a freely falling body is independent of its mass too, after all. Fig. 10 shows the results for the accelerations for two of the runs.

Fig. 10 shows the acceleration as a function of time of the projectile and CW for the conditions of column 2 and column 4 of Table II. The units of the lengths was in feet, and the masses in pounds, while the accelerations are in units of meter per second per second. The vertical line indicates the time at which the projectile leaves the sling. Note that one "g" is 9.8 m $s^{-2}$, so the CW sees a maximum of about 3.7 g's, and the projectile, about 40 g's for trebuchets having these proportions.

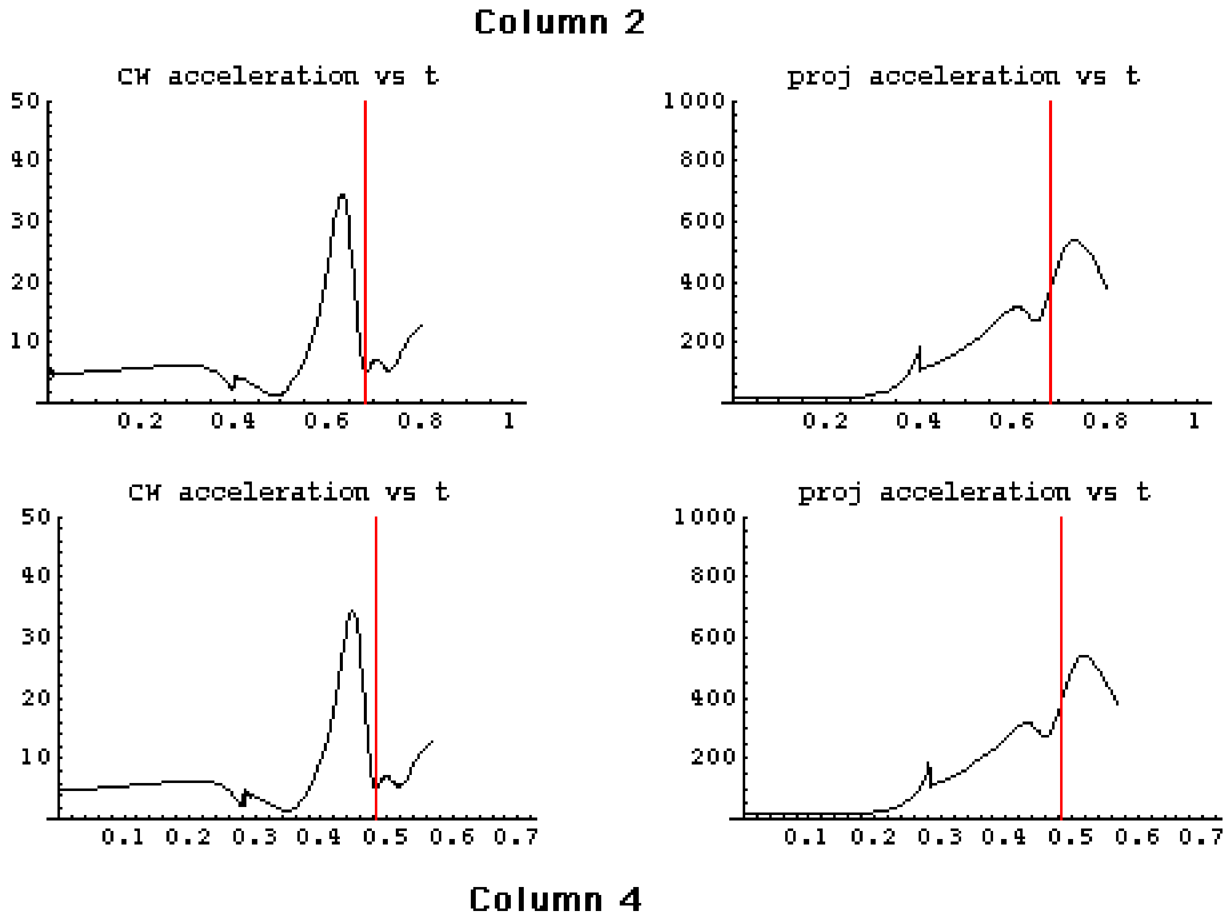



Fig. 10 The acceleration of the CW and the projectile as a function of time for two scaled trebs.

Fig. 10 documents very clearly that the accelerations for the two runs are in fact identical for all time, as expected from our simple argument. And, by the way, serves as a check on the correctness of the Mathematica simulation program.

While the accelerations are independent of the length, it is clear that the forces are not independent--they are proportional to the masses involved as well ( F = m a ). As an example, suppose we increase all of the lengths in our trebuchet by a factor of two. Then the volume of the CW box will increase by a factor of eight. Therefore, since the acceleration of the CW doesn't change, the force tending to break the CW bottom and its supports will also increase by the same factor of eight.

The strength of the hanger, with its doubled dimensions, will only increase by a factor of four--proportional to its increase in cross-sectional area. To "scale" the hanger to accommodate the increased load of a factor of eight, we need to increase its dimensions by $2\,2^{1/2} = 2.83$.

The strength of the axle would be increased by a factor of eight when its radius is doubled, but the span would be increased by a factor of two. Therefore, to scale the axle, we would increase its radius by 2 $2^{1/3} = 2.5$ to maintain the same factor of safety.

## Force on the axle

The force exerted on the axle during the throw is perhaps the most important quantity to obtain for assuring the structural integrity of the trebuchet. Judging by various anecdotal reports, the axle is the most likely part of the trebuchet to fail, and appears to be the component whose strength is most often overestimated in actual "eyeball" constructions.

To begin our analysis, we should make some qualitative observations about the physical situation. First of all, note that during the throw, the axle is primarily subject to a downward force exerted by the beam. The force comes mostly from the CW, transmitted by the hanger and beam, and, intuitively, we would expect it to be a maximum sometime around the bottoming out of the movement of the CW. The geometry near this configuration is as shown in Fig. 11. At this point, the hanger for the counterweight and the beam would clearly be in tension. The other end of the beam with the sling extended would clearly exert a force upward, mitigating to some extent the effect of the counterweight. Since the axle can experience and transmit forces in the horizontal and vertical directions we need to account for both components of the force, and a vector analysis simplifies things considerably.

The beam itself, and the force exerted on the axle due to its weight and rotation will be assumed to be negligible. The CW and its motion is clearly the main player here, and this is where we will start.

We note that the simulator (MacTreb* or WinTrebStar) does not give any information on the forces involved directly, so we need to deduce them from the solution obtained. The simulator gives the position of each of the parts of the trebuchet as a function of time, so we can easily calculate the velocity and acceleration of each of the points of interest. In particular, we can get the acceleration of the counterweight at each point in time, at least up to the point that the projectile leaves the sling. Our approach will be to illustrate the reasoning from a run for the trebuchet given in Table II.

Now consider the forces exerted on the CW that produced that acceleration. There are two. The first comes from gravity and is directed vertically downward and has a magnitude $m1\ g$. The other force comes from the hanger for the CW. Some assumptions were made in the simulator: the hanger, as well as the beam is modeled as a stiff rod; the hanger has negligible mass and is not loaded transversely, and the hanger joint is assumed to be frictionless. These assumptions require that the force exerted by the hanger, $F_h$, on the CW is directed along the hanger. In fact, it is a good check of the correctness of the solution obtained by the simulator, to verify that this is so. And our solution does meet this requirement, but that is another story....

This analysis allows us to deduce the forces on the axle from the forces on the projectile and CW, as shown in Fig. 11.

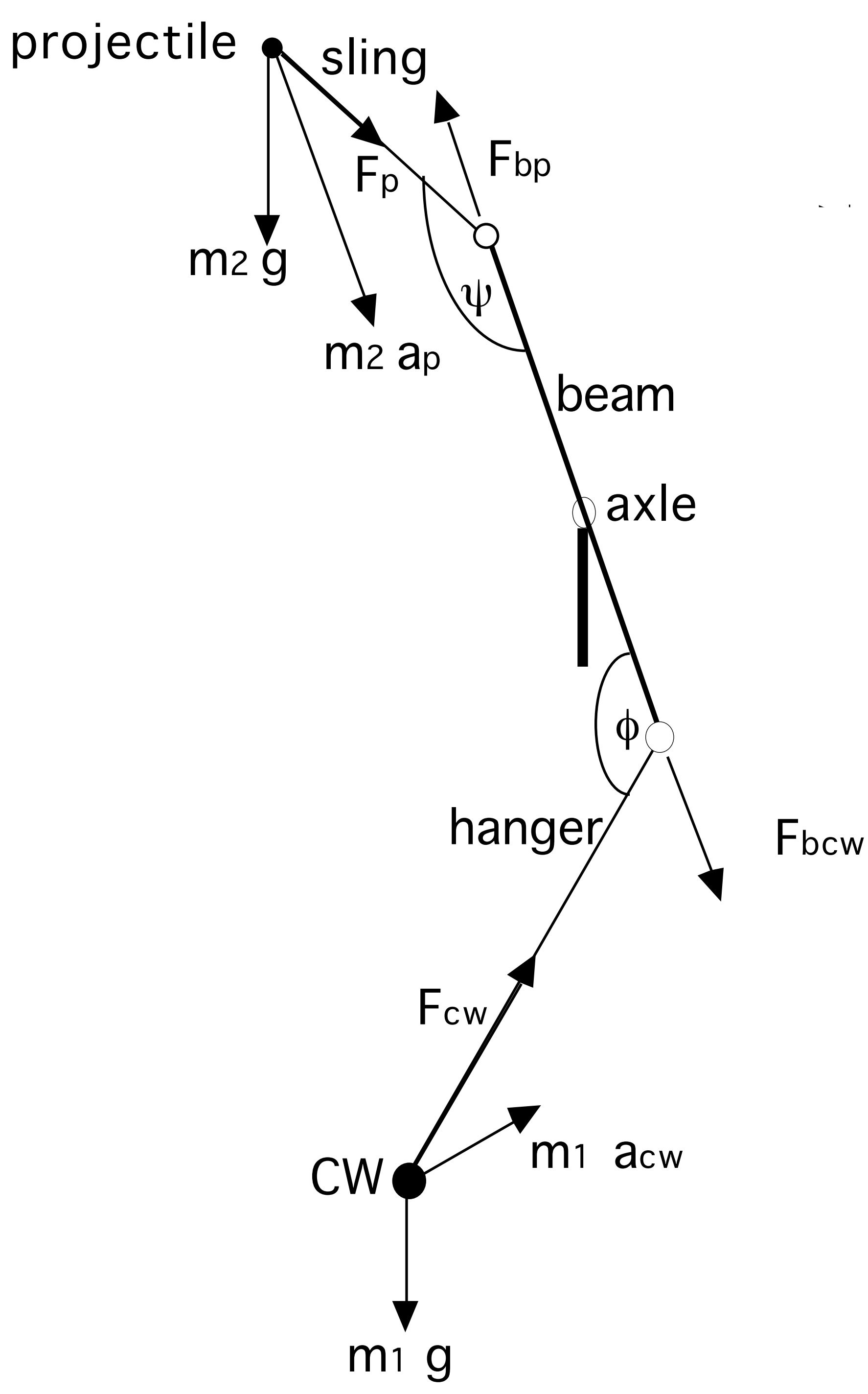



Fig. 11. The forces on the CW and the projectile when the beam is nearing the vertical position.

The resultant of the forces on the CW, $m_1 a_{cw}$ , is the sum of the force due to gravity and the force exerted on the CW by the hanger. In other words,

$$m_1 \vec{a}_{CW} = \vec{F}_{cw} + m_1\vec{g},$$

where the arrows signify vector quantities. Then the hanger force, directed along the hanger, can be resolved to a force along the beam as

$$\vec{F}_{bcw} = \vec{F}_{cw} \cos(\pi - \phi).$$

Similarly, the acceleration on the projectile is the resultant of the force from the sling and gravity:

$$m_2 \vec{a}_p = \vec{F}_p + m_2\vec{g},$$

which can also be resolved along the beam as

$$\vec{F}_{bp} = \vec{F}_p \cos(\pi - \psi).$$

The total force on the axle is the sum of the two forces directed along the beam:

$$\vec{F}_{bp} + \vec{F}_{bcw} = \vec{F}_{tot}$$

Since the masses, angles and accelerations (given by the simulation) are all known quantities, it is possible to get the force on the axle at all times. The magnitude of this force is, as usual,

$$F_{tot}= \sqrt{\vec{F}_{tot} . \vec{F}_{tot}}$$

which can be normalized (divided by) by the gravitational force on the CW, m1 g, to give a dimensionless quantity that is useful for application of these results to similarly proportioned trebuchets. That is,

$$F_n = \frac{F_{tot}}{m_1 g}$$

The total normalized force was calculated for the trebuchet described previously in the first column of Table II, using a suitably modified Mathematica program MathTreb 5.4, with the results as shown in Fig. 12.

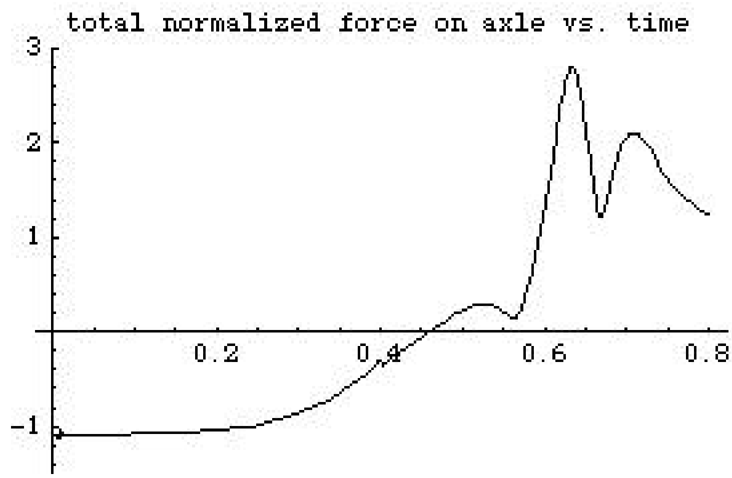



Fig. 12. The total normalized force on the axle for the nominal treb as a function of time.

The graph in Fig. 12 is drawn so that the unit on the y-axis is dimensionless, but from the way it is normalized, we can think of it as a graph of the number of g's on a mass of the CW. So, 2.0 on the y axis represents a force of twice the weight of the CW, but exerted on the axle. Thus, to get the force at any time for any treb with similar proportions at a point in time, multiply the y-axis value for that time by the mass times the acceleration in the corresponding units. Fig. 12 can therefore be used to predict the force on the axle for any treb having the proportions of the nominal trebuchet with parameters as shown in the first column of Table II . This procedure is best illustrated by a couple of examples.

Example 5: A big trebuchet

We want to build a treb proportioned like the nominal trebuchet, with l1= 1 m and m1 = 1600 kg. What is the maximum force on the axle? If the length of the axle is 1.5 m, how big should the radius of a wood axle be to support this? Assume the wood has a rupture modulus of 27 MPa.

Answer:

The maximum total normalized force read from the graph is about 2.8, so the maximum force on the axle during the throw would be 2.8 x 1600 kg x 9.8 m/s$^2$ = 43,900 N.

Use the flexural formula to get

$$r = ((3/16)\ P\ L/\sigma_{max})^{1/3}$$

and since $\sigma_{max}$ for wood is 27 x10$^6$ N/m^2,

$$r = ((3/16) \times 43900 \times 1.5/27{,}000{,}000)^{1/3}\ \text{m} = 0.077\ \text{m} = 7.7\ \text{cm}.$$

Thus, the required diameter for the axle is about 15 cm, without including any safety factor. If we want the strength of the axle to be four times this minimum, we would need an axle with a diameter of 15 x $4^{1/3}$ = 24 cm.

Example 6: Calculation of Force on the Axle and its required diameter. Suppose we have a treb with a short beam length of l1= 1 ft, and l2 = 4 ft, l3 = 3.5 ft, l4 = 1 ft, m1 = 80 lbs, m2= 0.8 lbs and mb = 6.75 lbs and the beam is initially at 45° with respect to the ground plane. The axle is 1 foot long. What is the maximum magnitude of the force on the axle? Will an axle 0.5 inches in diameter made of mild steel be strong enough? Would a rod of wood 1 inch in diameter break?

## Answer:

Since the proportions of this treb is the same as that used to construct Fig. 12, and since the acceleration is independent of scale, we can use the total normalized force graph. The force is normalized by m1 g, so if we want to convert to an equivalent weight, we multiply it by m1. From the figure, we see the maximum normalized force on the axle is about 2.8. Then the force is 2.6 x 80 pounds = 224 pounds. Note that we should not multiply by the acceleration of gravity in American units (32 f $s^{-2}$) here, because the pound is a unit of force, and already includes it, sort of. One more reason the metric system is easier to understand!

Use the flexural formula to give

$$\sigma_{max} = (3/16)\ P\ L\ /r^3 = (3/16) \times 224 \times 12/(0.25^3) = 32\ 000\ p/in^2.$$

The rupture strength of mild steel is 36,000 p/in$^2$, which means that the axle should be OK, but with little margin for error. If you plan to double your CW and projectile weight sometime, you'd be on dangerous ground....

For the 1 inch diameter wood axle, we get

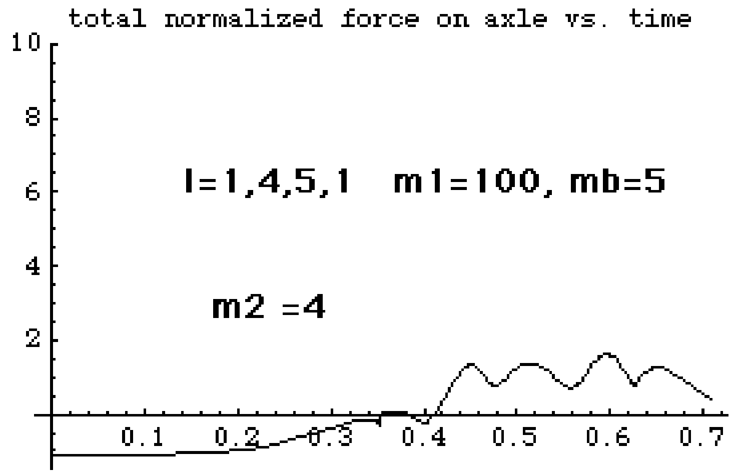
$$\sigma_{max} = (3/16) \times 224 \times 12/(0.5^3) = 4000\ p/in^2$$

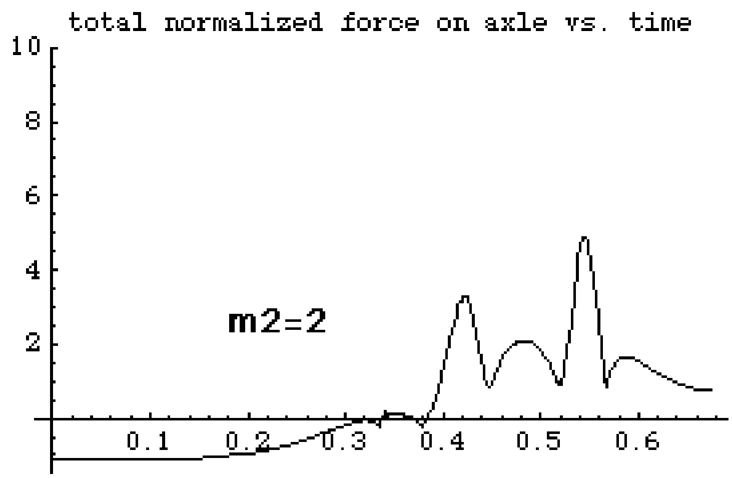
Since the strength of wood in compression is about 4000 p/ in$^2$, we would have no safety factor. Watch out below!

## Other Trebuchets

If one is not going to build a treb with the proportions identical to the "nominal" treb (where is the fun in that?), then we cannot use the normalized force graph and we need to find another path to discovering the forces involved. One possible approach is to look at a wide variety of configurations and attempt to find some generalizations and rules of thumb.

One important rule is to note the very harmful effect of decreasing the mass of the projectile relative to the mass of the CW. We have noted that the projectile acts on the axle in a direction opposite to that of the CW, decreasing the force on the axle. Fig. 13 shows three graphs of the normalized force on the axle for similar trebs with decreasing projectile mass, m2.


total normalized force on axle vs. time
10
8
6
4
2
0.1 0.2 0.3 0.4 0.5 0.6 0.7
l=1,4,5,1 m1=100, mb=5
m2 =4


total normalized force on axle vs. time
10
8
6
4
2
0.1 0.2 0.3 0.4 0.5 0.6
m2=2

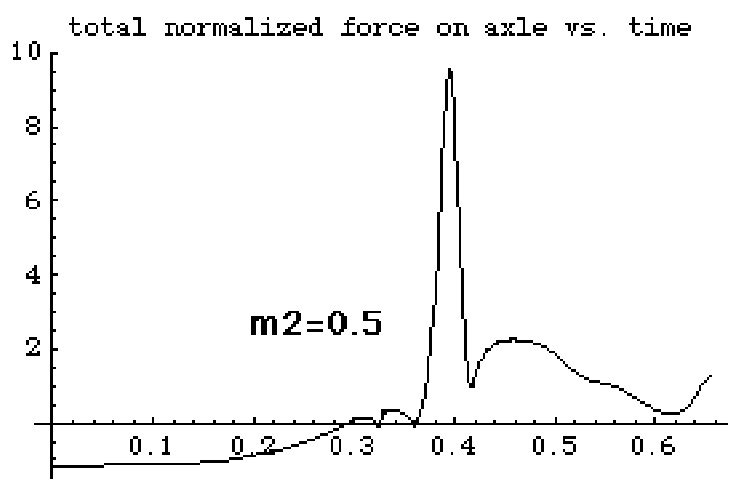

total normalized force on axle vs. time
10
8
6
4
2
0.1 0.2 0.3 0.4 0.5 0.6
m2=0.5

Fig 13. The total normalized force on the axle as a function of time for decreasing projectile masses, m2.

This third graph, with m2 = 0.5, has a very sharp maximum of normalized force, reaching almost 10 g's. This maximum becomes largest when the mass of the projectile is smallest. When the mass of the projectile is negligible, of course, the mass of the pouch and sling prevents the mass on the end of the beam from being zero.

This chain of reasoning shows in a forceful way why we should never "dry fire" the treb, and why we should not plan on trying to throw a pea across the channel. It also shows why the throws that misfire are most harmful to the integrity of the trebuchet. Misfires produce the maximum force on the axle, and on the other parts as well.

It is worth noting here some limitations of the simulator in predicting these forces. The main limitation is that all of the parts of the treb are assumed to be infinitely stiff. When the real parts of the trebuchet stretch and bend during the throw, the sharpness and magnitude of the forces are decreased below the predictions shown in the normalized force graphs. How much are they decreased? It is hard to generalize, but surely we would expect the sharp spikes as shown in Fig. 13 to be over-estimates of the magnitude of the force that needs to be protected against, and under-estimates of the time over which it is exerted.

After looking at many simulations for different configurations of the trebuchet, taking the normalized force diagrams as the over-estimates that they are, one might reasonably conclude that something like 10 g's would surely be a sufficiently large acceleration to protect against, and should work for most designs.

Another method for examining the various forces for different designs is to use the free Mathematica program "Trebfornet" available at the "Algorithmic Beauty of Trebuchets" site. This requires the rather expensive Mathematica (Wolfram) program to run, and is not for everyone. It is easy, however, to modify to get graphs of all the forces and so on, and is the flexible path to a complete mastery of novel approaches and designs.

## Sideward force on the axle

The force on the axle is not just in the downward direction--there is a sideward force as well. This also needs to be protected against. The graphs in Fig. 14 shows the total normalized force from Fig. 12 (for our nominal treb described in the first column of Table II) broken into its two components in the vertical and horizontal directions. The vertical force is always downward, but the horizontal force first goes to the left, then the right. It is interesting to note that the horizontal force is quite sizable in this particular treb. Evidently the axle would have to be pretty well braced against horizontal movement by the a-frame holding it.

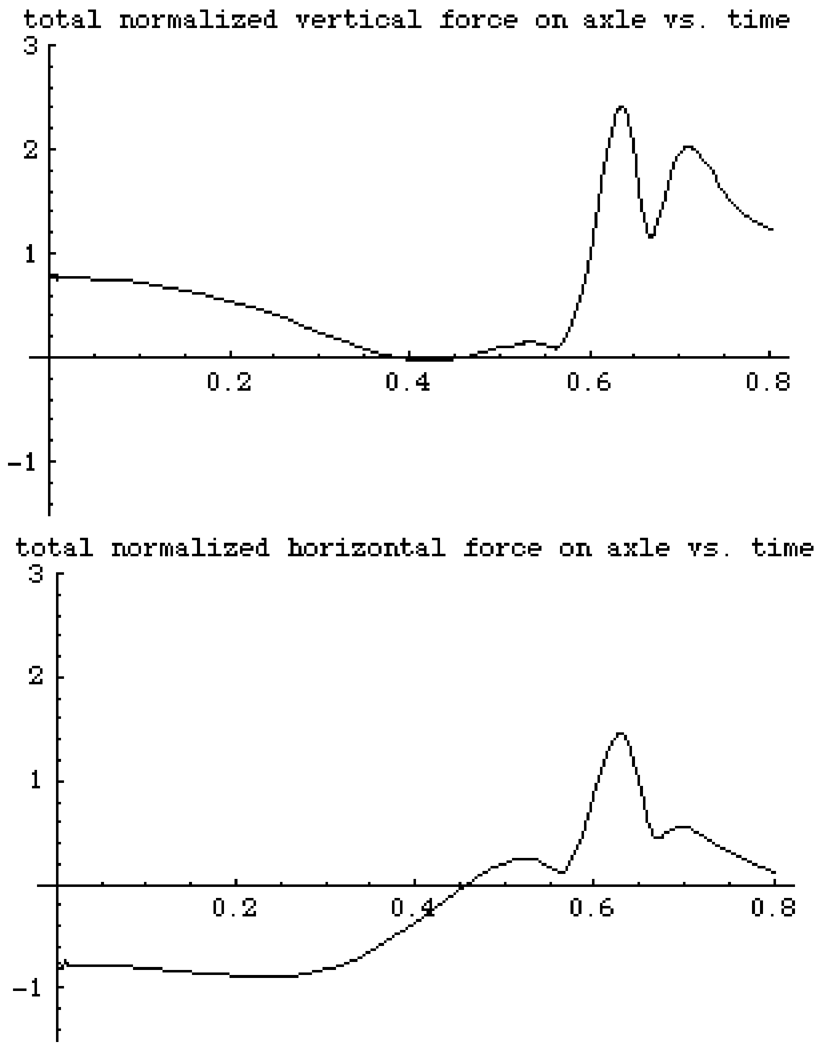



Fig. 14. The total normalized vertical and horizontal forces on the axle of the nominal treb.

## Forces in the Truss

The axle is normally supported by a framework that can take on a rather wide number of forms, and we can only outline the solution for a particular embodiment here. Other configurations can be readily solved for if one understands this particular example, and follows a similar procedure. The method is to first determine the reaction forces (the external forces keeping the truss in equilibrium), then each of the unknown forces in each of the struts by isolating each node in turn.

We can calculate the forces in the truss that supports the axle by referring to Fig. 15.

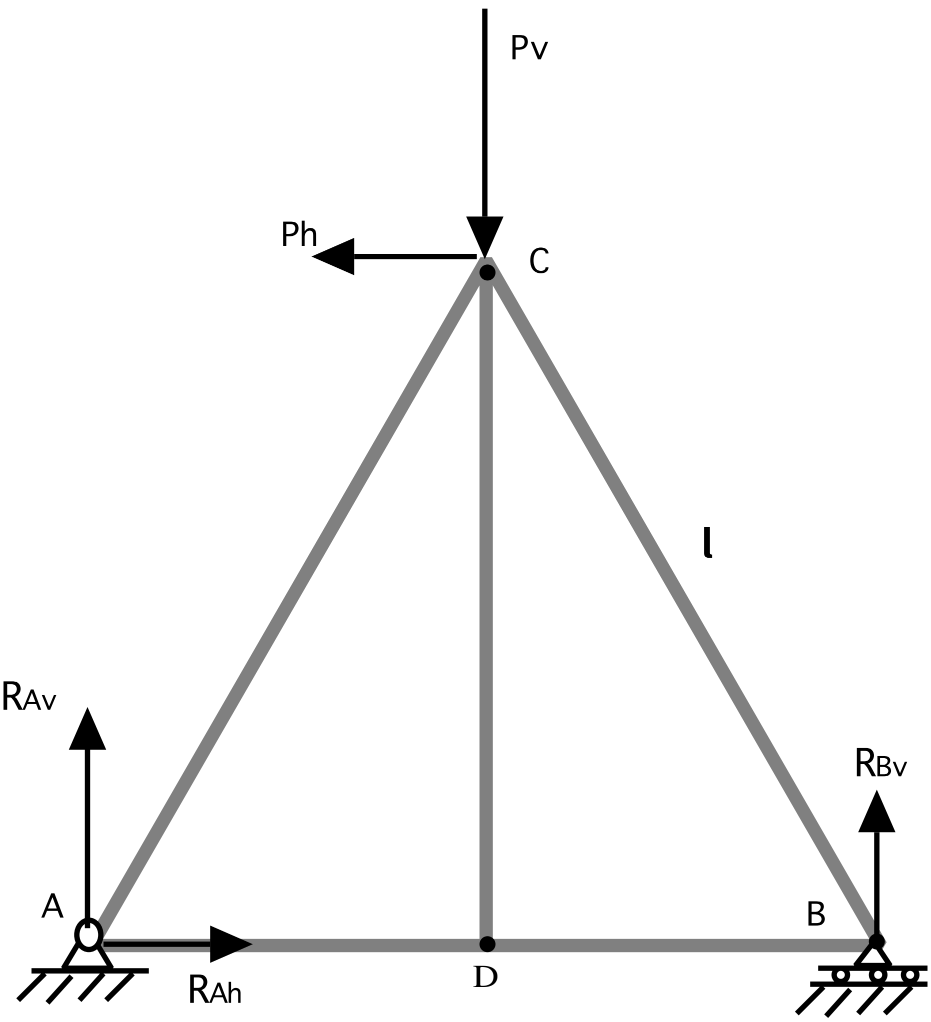



Fig 15. A diagram of the truss supporting the axle, showing the reaction forces, R, and the imposed force P.

The angle between the sides of the truss is particularized here to be 60°, and the sides have length l. It is very easy to modify the calculations for other angles and proportions, and is left as an exercise for the reader.

Here we idealize the truss supporting the axle as consisting of paired equilateral triangles with sides of length l. The three sides of the triangle are taken to be uniform beams with a rectangular cross-section. The force down on the truss is shown as Fv, which is one-half of the downward force on the axle, which we have just calculated. We will look at the general case, where the sideward force is Fh and is non-zero. We assume, to a first approximation, that the beams are weightless and can transmit only longitudinal forces of either compression or tension. They are pinned together at each of the corners. As normally done (to make the structure determinant), we assume the node at the left (designated "A") is connected to the ground with a hinge, and on the right with a slider or roller. The hinge at "A" can sustain forces in both the x and y directions, but the roller can only exert a force in the vertical direction.

The first step in getting the forces in the truss beams is to get the reaction forces on the truss as a whole. Since the whole structure is in equilibrium, the sum of the forces in the x direction is zero, the sum of the forces in the y direction is zero, and the sum of the moments about any point is zero.

Using $\Sigma$ Fx = 0 gives

$$R_{Ah} - P_h = 0,$$

or

$$R_{Ah} = P_h.$$

That is, the horizontal reaction force is directed to the right, as drawn.

Similarly, $\Sigma$ Fy = 0 yields

$$Pv = Rav + Rcv$$

The moments of the forces about vertex A sum to zero, to give

$$l\, R_{Bv} - \frac{l}{2} P_v + \frac{l\sqrt{3}}{2} P_h = 0$$

so

$$R_{Bv} = \frac{P_v - \sqrt{3}\, P_h}{2}$$

and

$$R_{Av} = \frac{P_v + \sqrt{3}\, P_h}{2}.$$

The reaction forces are supplied by the support (the ground) for the truss, and so are usually of little interest in themselves. Their importance is found in finding the forces in the struts. We start with the joint at "A" and draw the free body diagrams, in which a joint is isolated, and all of the known forces are drawn as vectors, and the unknown forces are shown acting along a known direction. We assume all the unknown forces are in tension, and so if the sign for one of the unknowns turns out to be negative after calculating it, we know that the strut is in compression. Our object is to systematically get the magnitude of all of the unknown forces in the struts. We denote the unknown forces by the two letters of the connecting nodes.

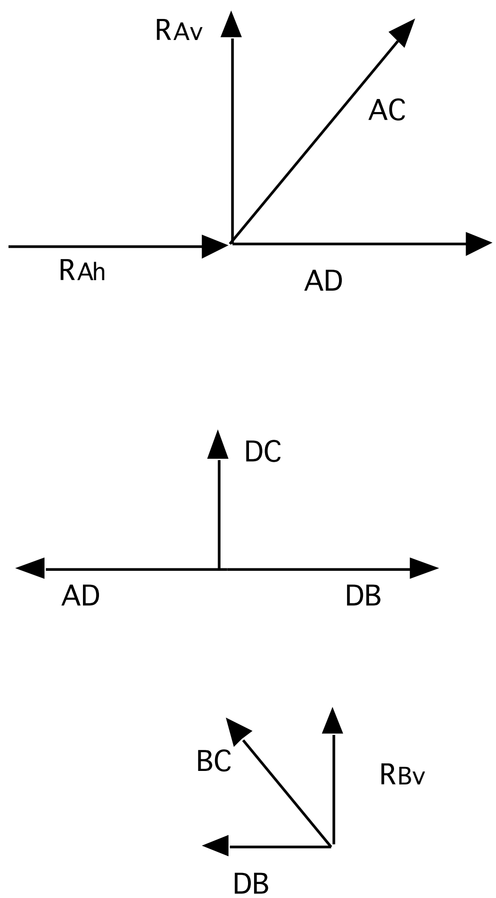



Fig. 16. The free body diagrams for the three nodes, A, D and B.

Using $\Sigma$ Fx = 0 on the first diagram of Fig. 16. gives

$$AC \cos 60^\circ + AD + R_{Ah} = 0$$

and $\Sigma$ Fy = 0 yields

$$AC \sin 60^\circ + Rav = 0,$$

so we can solve for the forces AD and AC to get

$$AC = -(Ph + Pv/\sqrt{3}),$$

showing AC is in compression when Ph and Pv are in the direction drawn, and

$$AD = Pv/(2\sqrt{3}) - Ph/2.$$

Moving to node D, using $\Sigma Fx = 0$ on the second free body diagram of Fig. 16 gives $DB = -AD$ and using $\Sigma Fy = 0$ gives $DC = 0$.

The node at B can now be solved to give

$$-BC \cos 60^\circ - DB = 0$$

or

$$BC = -2\ DB = 2\ AD = -(Ph - Pv/\sqrt{3}).$$

The resulting forces in the struts can be summarized as shown in Fig. 17.

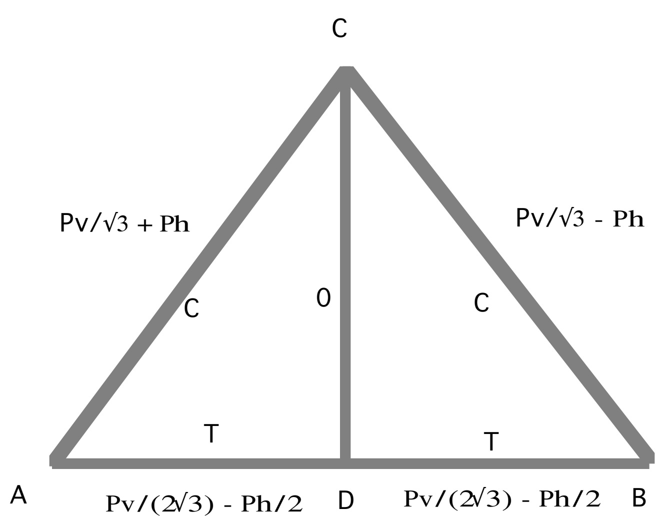



Fig. 17. The forces in the struts. Here C signifies that the member is in compression, T signifies tension.

## Example 7: Force on diagonal strut

For the case shown in Fig 14, using the values for the "nominal" treb described in the second column of Table I, with the mass of the CW taken to be 1000 pounds, is a 2 x 4 strong enough?

Solution:

For a CW of 1000 pounds, Fig. 14 shows a maximum vertical force on the axle of about 2600 pounds and a maximum horizontal force at the same time of about 1500 pounds at the same time. This is divided into two trusses, so the maximum force in the diagonal strut is calculated to be about 1500 pounds. As shown in a preceding example, the area of a "2x4" is about 5.25 $in^2$, so the maximum compressive force is about 300 pounds $in^{-2}$. The rupture strength for eastern white pine is 8600 psi. If it is second grade, it can withstand about 45% of this, or about 4000 psi. The 2x4 is plenty strong enough for a normal throw.

If there is a misfire, as discussed previously, the forces can be as much as 10 times greater than this, however. The 2x4 would then be subject to about 3000 psi, and there would not be much of a safety factor. Better use something bigger. Always plan for a misfire!

## Force on the Sling

The force tending to break the sling is Fp, which is available from the simulation. For the treb discussed above, it is shown in Fig. 18 normalized by the

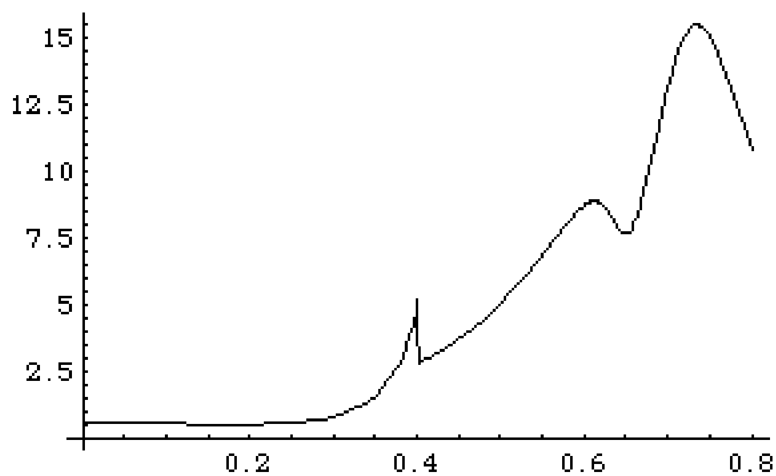



Fig. 18. The normalized force tending to break the sling of the nominal treb.

weight of the projectile, m2 g, The spike shown at about 0.4 s is a numerical artifact of the method used in obtaining the solution and can be ignored. This graph can be used in a manner identical to that discussed in connection with Fig. 12. For this configuration, we see that the maximum force that the sling needs to be able to withstand is about 15 g's on the projectile's mass. If the mass is 8 pounds, then the force tending to break the sling is about 120 pounds.

Here is one way we can really increase the efficiency of the trebuchet. One has a tendency to make the sling much stronger than it needs to be, and together with the pouch, much heavier than it needs to be. If the mass of the pouch and sling is 20% of the mass of the projectile, say, then the projectile will have a range decreased by that same 20%.

This calculation is probably pretty accurate, especially if the rope is strong and made out of a material like polypropylene, which is stiff compared to nylon.

## Forces on the sling release finger

We now discuss, by way of an example, the forces on the prong or finger that one end of the sling is looped over. There are many ways to construct this part of the trebuchet, limited only by the ingenuity of the designer, and so our discussion is limited to this one embodiment. The diagram in Fig. 19 shows the prong attached to a base plate which is bolted to the end of the beam. The prong is made from a bolt threaded through two nuts which are welded to the base plate. The prong has been bent at an angle $\delta$ with respect to the axis of the beam. The right end of the carriage bolt is unthreaded and smooth, with the head cut off. A washer tied to the end of the sling slides along the smooth part of the bolt during the movement of the beam, and so the sling exerts a variable force at some angle to it during the throw. Eventually the force along the finger is large enough that the washer slides off of the end of the bolt, releasing the sling.

The bending angle and length of the prong control the exact moment of release of the sling. The prong will turn during the release unless a lock nut (not shown) is used to prevent its movement; this is placed next to the nut nearest the end of the beam. By making the bolt removable by unscrewing it, one allows for experimentation with different angles and lengths, and the release point can be adjusted to the optimum.

We want to make an estimate of how large a force to allow for, how much the finger can withstand, and what the forces are that the welds have to withstand.

The force Fs is variable in magnitude and direction, and is the same one discussed in the previous section as the force tending to break the sling.
We need only concern ourselves with the maximum of this force, and, we will assume that it is perpendicular to the finger. Thus we are over-estimating the force by a bit, which is a conservative thing to do.

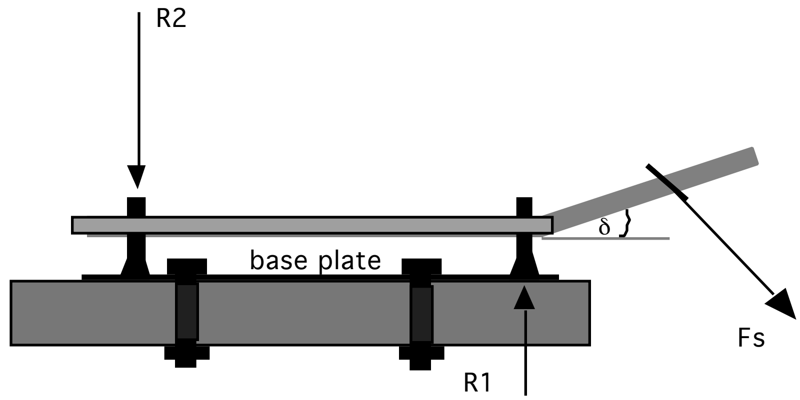



Fig. 19. A configuration for the sling release finger and the forces acting on it.

The maximum bending moment at x, where x is the distance from the rightmost nut is clearly Fs (L-x), where L is the length of the prong extending to the right of that supporting nut. The maximum stress is at x = 0 and so the flexure formula gives the maximum stress in the prong as

$$\sigma = Fs\ L\ r/I = 4\ Fs\ L/(\pi\ r^3),$$

where r is the radius of the prong.

Example 8: Steel prong on a big trebuchet
Suppose we build the trebuchet like in the previous example of the "big" trebuchet, with l1 = 1 m and m1 = 1600 kg, and the mass of the projectile is 1% of this. Would a steel prong 1/4 inch in diameter 4 inches long be strong enough?

Answer:
The normalized force, as discussed in connection with the example of the sling strength, would be about 15 g's, so the force would be Fs = 16 kg*15*9.8 m $s^{-2}$ = 2352 N = 528 pounds. The maximum stress is then

$$\sigma = 4*528*4/(\pi\ 1/8^3) = 1.3\ X10^6 \text{ psi.}$$

This is a factor of 36 larger than the yield strength of steel, so the diameter would have to be about 3 times as big, or a little more than 3/4 of an inch.

The force on the left-hand bolt tending to detatch it from its support is readily calculated by taking moments about the right hand bold. An over-estimate is obtained by again assuming that Fs is normal to the prong, acting at a distance L from the right-hand nut. Equating the moments gives

$$R_2 = Fs\ L/Ln,$$

where Ln is the distance between the nuts.

The weld holding the nut to its support has a configuration that we can idealize as shown in Fig. 20. The weld is weakest along a line as shown and has a depth w. If the length of the weld on each side of the nut is t (thickness of the nut), then the total area of the weld is 2 t w. The stress in the weld is therefore

$$\sigma = R_2/(2\ t\ w) = Fs\ L/(2\ t\ w\ Ln)$$

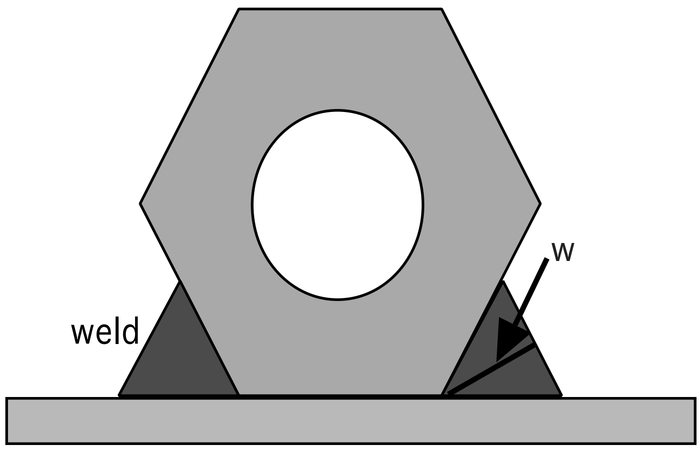



Fig. 20 The geometry of the weld holding the nut to the prong base-plate.

A good weld will be almost as strong as the steel itself, so we can use 36000 psi as an estimate of its yield strength.

Example 9: Weld strength
Suppose the nut is one inch thick. Will the weld fail for the trebuchet described in the last example? Use Ln = 5 in and w = 1/4 in.

Answer:
Plug in the numbers:

$$\sigma = 528 \times 4/(2 \times 1 \times 0.25 \times 5) = 840 \text{ psi.}$$

This is substantially less than the yield strength of steel. It is plenty strong enough.

Conclusions
Some relatively simple considerations can go a long way toward making a safe trebuchet design that won't self-destruct the first time it is fired. The calculation for the required dimension of the axle is quite easy, especially if one uses the rule of thumb that the maximum force that needs to be designed for is ten times the weight of the counterweight. The analysis of the forces exerted as a function of the projectile mass clearly show the very harmful effects of misfires and dry-firing on the integrity of the trebuchet.

The requirements for the beam, counterweight box, sling and its release prong are also readily found once the maximum accelerations of the CW and projectile are known from a simulation. Dimensional analysis shows that the accelerations of scaled trebuchets are invariant to size, which leads to a great simplification in the analysis of the forces acting on them. The internal forces in the struts of the truss supporting the axle are also derived for a simple configuration.

These considerations could no doubt be made more elaborately using finite element analysis or computer aided design programs, and can be profitably pursued by the cognoscenti. The analysis presented here, while more limited in scope and reliability, should be sufficient for the design problems faced by the enthusiast.

# Appendix I

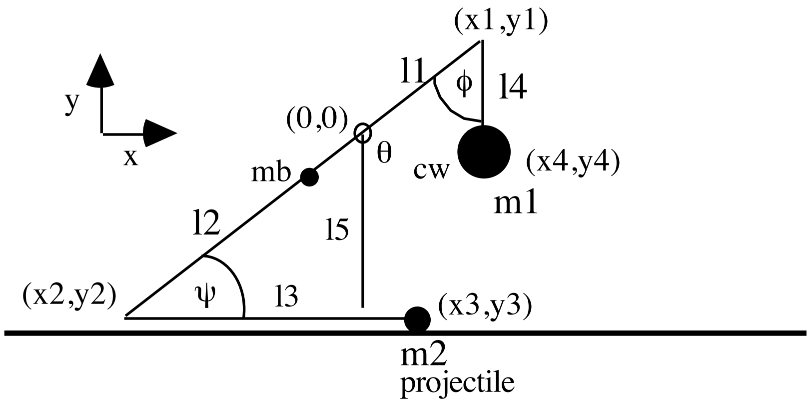



Fig. A1. The geometry of the trebuchet, showing the three angles taken as the independent variables, in a configuration at the start of the movement. Here mb is the mass of the beam. The origin of the coordinate system is at the axle, and the positive y direction is up.

# Appendix II

A Mathematica program for solving for the shape of the uniform beam under a load.

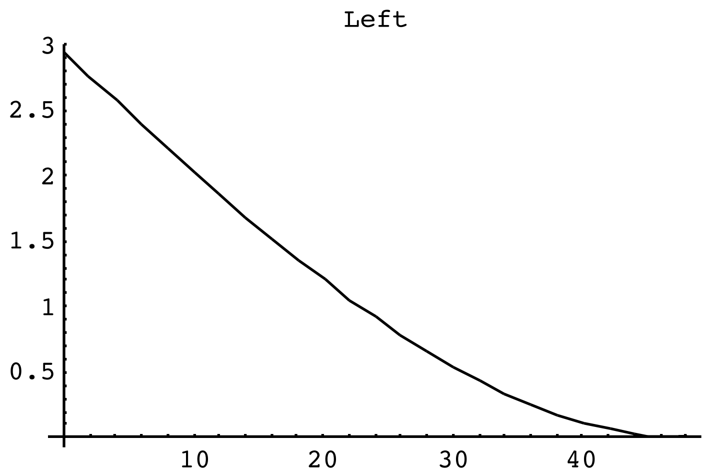
```
(*Solve for the bending of the beam of a treb*)
(* let k= Y I; m = m1 g *)
(* set up the Differential equation for the uniform beam*)
eqlft=k* D[y[x],{x,2}]==m l1/l2 x
(*solve the DE*)
slft=DSolve[{eqlft,y[l2]==0,y'[l2]==0},y[x],x];
ylft[x_]=y[x]/.Together[Flatten[slft]]

(* Similarly solve for the right end's deflection*)
eqrt=k* D[y[x],{x,2}]==m (l1/l2)* x- m*(1+l1/l2)*(x-l2)
srt=DSolve[{eqrt,y[l2]==0,y'[l2]==0},y[x],x];
yrt[x_]=y[x]/.Together[Flatten[srt]]

(* plot the left end solution*)
Plot[ylft[x]/.{l1->12,l2->48,m->500,k->.98*1.6*10^6},{x,0,48},
  PlotLabel->"Left"];
Plot[yrt[x]/.{l1->12,l2->48,m->500,k->.98*1.6*10^6},{x,48,60},
  PlotLabel->"right"];
```

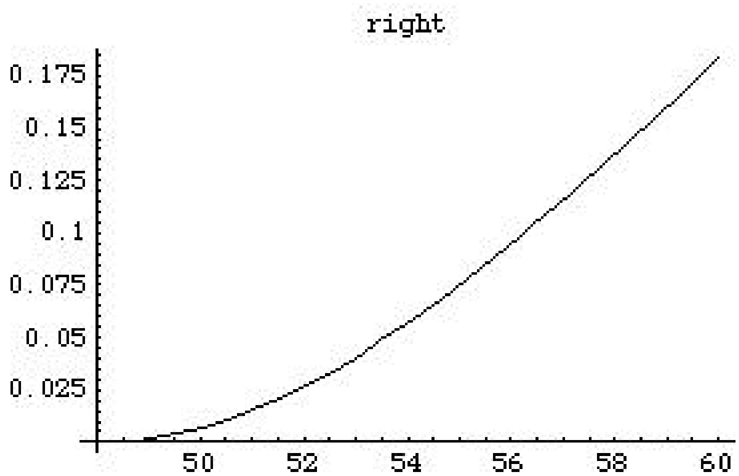
$$k\, y''[x] == \frac{l1\, m\, x}{l2}$$

$$\frac{l1\, l2^2\, m}{3\, k} - \frac{l1\, l2\, m\, x}{2\, k} + \frac{l1\, m\, x^3}{6\, k\, l2}$$

$$k\, y''[x] == \frac{l1\, m\, x}{l2} - \left(1 + \frac{l1}{l2}\right) m\, (-l2 + x)$$

$$-\frac{-3\, l1\, l2^2\, m - l2^3\, m}{6\, k} - \frac{(2\, l1\, l2\, m + l2^2\, m)\, x}{2\, k} + \frac{(l1 + l2)\, m\, x^2}{2\, k} - \frac{m\, x^3}{6\, k}$$


Left
3
2.5
2
1.5
1
0.5
10
20
30
40


right
0.175
0.15
0.125
0.1
0.075
0.05
0.025
50
52
54
56
58
60

## References

For a general simulation program use WinTrebStar or MacTreb*, which can be downloaded from the "Algorithmic Beauty of Trebuchets" site at www.members.home.com/dimona. Written by the author of this document, it can be used for determining the efficiency of a trebuchet design, and for experimenting with trade-offs and optimization of parameters.

"The Mechanics of the Trebuchet" is a companion article to this one, which describes the assumptions and methods used to derive the algorithm used in these programs. It also offers some general rules for efficient trebuchets, a discussion of scaling and dimensional analysis, and some observations on the requirements for the sling release mechanism.

A good introduction to the strength of materials as used in this document is at http://physics.uwstout.edu/strength/indexfbt.htm#TABLE%20OF%20CONTENTS. See especially chapters 4, 5 and 7.

Another excellent tutorial on strength of materials is at http://www.lafayette.edu/kayserj/statics/cover.htm#index.

The mechanical properties of wood in all of its glory is described in the Forest Products Laboratory 1999 Wood Handbook, which is at http://www.fpl.fs.fed.us/documnts/FPLGTR/fplgtr113/fplgtr113.htm Includes extensive tables, definitions, grading, and stress formulas for commonly encountered geometries.